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Katrin Wehrheim

Uhlenbeck Compactness

European Mathematical Society

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Preface

The origin of this book lies at the beginning of my graduate studies, when I just could not understand Uhlenbeck compactness, let alone see whether it would also hold for my cases – on manifolds with boundary. There seemed to be certain gaps between standard mathematics education and the analytic background needed to understand a very fundamental research paper in Yang-Mills theory.

A first gap was closed while I was working with Laurent Lazzarini – we finally understood the L^p-estimate for the operator $\mathrm{d}\oplus\mathrm{d}^*$. From there I went on through the depth of Uhlenbeck compactness - guided and encouraged by my supervisor, Dietmar Salamon, and always keeping my boundaries in mind. I had to overcome some obstacles, found a few subtleties, and finally arrived at a detailed understanding of Uhlenbeck compactness and its (not far from obvious) generalizations to manifolds with boundaries and manifolds exhausted by compact sets. All this work seemed to be worth writing down, so I ended up writing the book that I would have needed at the beginning of my graduate studies: A selfcontained exposition of Uhlenbeck compactness with all the analytic details, which only refers back to standard textbooks for classical results.

After having difficulties in finding references on L^p-results for the Neumann boundary value problem I included a preliminary part on that topic. This provides the required results for the Neumann problem and also shows the general philosophy behind elliptic boundary value problems – it taught me the deep truth in Uhlenbeck's sentence "Elliptic systems are well-behaved on Sobolev spaces".

So this book intends to be a guide to students on their way into the analysis of Yang-Mills theory. I also hope that it will be a useful reference for those who need to know about a particular detail in or behind Uhlenbeck compactness.

Let me stress that I do not claim any original work. The minor generalizations of Uhlenbeck compactness that are stated in this book have been known before. However, there are a lot of details and alternative approaches to certain parts of the proofs that probably cannot be found elsewhere. These bits and pieces are specified in the introduction.

Much credit for this book goes to Dietmar Salamon - for the idea as a start, for all the help with obstacles, but also for teaching me how to overcome them myself and finally, how to write mathematics. I'm also glad to have this opportunity to thank the 'symplectic gang' in and around the ETH Zürich for a great working atmosphere, stimulating discussions, and some glorious symplectic action!

Contents

Introduction

The aim of this book is to explain in full detail the proofs of Uhlenbeck's weak and strong compactness theorems. They were proven by Karen Uhlenbeck in 1982, c.f. [U1, U2]. Another textbook reference is [DK] by Donaldson and Kronheimer. Uhlenbeck's compactness results play a fundamental role in gauge theory. The strong and weak compactness theorems both concern sequences of connections on principal bundles with compact structure groups. The strong compactness theorem deals with Yang-Mills connections whereas in the weak compactness theorem the connections are not required to satisfy any equation.

An elementary observation in gauge theory is that the moduli space of flat connections over a compact manifold with a compact structure group is compact in the $\mathcal{C}^\infty$-topology. This is obvious from the fact that the gauge equivalence classes of flat connections are in one-to-one correspondence with conjugacy classes of representations of the fundamental group. (Here the bundle is not fixed but rather is also determined by the representation.) The weak Uhlenbeck compactness theorem is a remarkable generalization of this result. It asserts, in particular, that every sequence of connections with uniformly bounded curvature is gauge equivalent to a sequence, which has a weakly $W^{1,p}$-convergent subsequence (for any fixed p). In the case of abelian groups the proof reduces to Hodge theory, but in the nonabelian case it is highly nontrivial. This theorem lies at the heart of the compactness results for many equations in nonabelian gauge theory, such as the Yang-Mills equations, the vortex equations, or the rank two Seiberg-Witten monopole equations.

Uhlenbeck's strong compactness theorem asserts that every sequence of Yang-Mills connections with uniformly bounded curvature is gauge equivalent to a sequence, which has a $\mathcal{C}^\infty$-convergent subsequence. This result can be reduced to the weak compactness theorem as follows: The weak limit is again a Yang-Mills connection and hence is gauge equivalent to a smooth connection. Now one can put the sequence into relative Coulomb gauge with respect to the limit connection. Then $\mathcal{C}^\infty$-convergence follows from the fact that the Yang-Mills equation together with the gauge fixing condition form an elliptic system. By the same argument one obtains similar compactness results for all gauge theoretic equations that together with the relative Coulomb gauge form an elliptic system.

An important application of Uhlenbeck's theorem is the compactification of the moduli space of anti-self-dual instantons over a four-manifold. These compactified moduli spaces are the central ingredients in the construction of the Donaldson invariants of smooth four-manifolds [D2] and of the instanton Floer homology groups of three-manifolds [F]. Anti-self-dual instantons are special first order solutions of the Yang-Mills equation. Uhlenbeck's theorem asserts that noncompactness can only occur in sequences with unbounded curvature. In this case a conformal rescaling argument shows that instantons on the four-sphere bubble off. For a suitably chosen subsequence bubbling only occurs at finitely many points, and on the complement one has $\mathcal{C}^\infty$-convergence. Now Uhlenbeck's removable singularity theorem [U1] guarantees that the limit connection extends over the four-manifold. In the case of simply connected four-manifolds with negative definite intersection forms Donaldson used these compactified moduli spaces to prove his famous theorem about the diagonizability of intersection forms [D1].

Next we shall discuss the results proved in this book in more detail. A gauge invariant measure for the curvature is the L^p-energy of a connection,

$$\mathcal{E}(A) = \int |F_A|^p.$$

This energy is conformally invariant for $p = \frac{n}{2}$ on an n-manifold. As a consequence, for $p \leq \frac{n}{2}$ the moduli spaces of connections with bounded energy are not even compact in the L^1-topology.[1] For $p > \frac{n}{2}$, however, the weak Uhlenbeck compactness theorem asserts the compactness of these moduli spaces in the weak $W^{1,p}$-topology. The strong Uhlenbeck compactness theorem asserts the $\mathcal{C}^\infty$-compactness of the moduli spaces of Yang-Mills connections with bounded L^p-energy for $p > \frac{n}{2}$. Again this fails for $p = \frac{n}{2}$. Explicit examples in the case $n = 4$ follow from the ADHM construction [ADHM] of anti-self-dual instantons on the four-sphere with fixed L^2-energy. For example, the gauge equivalence classes of anti-self-dual $SU(2)$-connections over S^4 with L^2-energy $8\pi^2$ are parametrized by $\mathbb{R}^4 \times \mathbb{R}^+$ [DK, 3.4.1].

The weak and strong Uhlenbeck compactness theorems were originally stated for closed base manifolds with a fixed metric, but they generalize to several other situations. In this book we directly prove the Uhlenbeck compactness theorems for compact manifolds with boundary, as stated in theorems A and E. For the strong Uhlenbeck compactness this means that we consider the Yang-Mills equation with boundary condition $*F_A|_{\partial M} = 0$. Furthermore, there are generalizations to varying metrics and to manifolds that are exhausted by compact deformation retracts, as stated precisely in theorems A$'$ and E$'$.

These generalizations are needed in the following applications: The proof of the metric independence of the Donaldson invariants requires the compactification of parametrized moduli spaces. These contain pairs consisting of a metric (in a

[1] Let A be a connection of finite energy on the trivial bundle over $S^4 \cong \mathbb{R}^4 \cup \{\infty\}$ and consider the rescaled connections $A_\sigma(x) = \sigma A(\sigma x)$, then $\mathcal{E}(A_\sigma) = \sigma^{2p-n}\mathcal{E}(A)$ is bounded as $\sigma \to \infty$ but the pointwise norm of A_σ converges to a multiple of the Dirac distribution (by theorem C.5).

fixed path between two metrics) and a Yang-Mills connection with respect to that metric. So one has to allow for the metric to vary along with the sequence of connections in the strong Uhlenbeck compactness theorem. One can, however, a priori choose the sequence such that the metrics converge. The strong Uhlenbeck compactness also is frequently used for two types of noncompact base manifolds. Firstly, manifolds with finitely many punctures arise from the bubbling off analysis in two ways: One has convergence on a compact base manifold with finitely many punctures, and the bubbling off analysis near every puncture yields a sequence of connections on larger and larger balls exhausting $\mathbb{R}^4$. Secondly, one considers manifolds with cylindrical ends, for example, in order to define Donaldson invariants for manifolds with boundary. In that case one glues a cylindrical end to each boundary component. In Floer theory the product of the real line with a three-manifold occurs naturally as the domain of the gradient flow lines of the Chern–Simons functional.

Both generalizations, theorems A′ and E′, are proven by an extension argument of Donaldson and Kronheimer. This requires the restriction to manifolds that are exhausted by compact deformation retracts to ensure that gauge transformations on the compact sets can be extended to the whole manifold. Note that this covers both the case of manifolds with punctures and of manifolds with cylindrical ends. Hence these generalizations suffice for all the applications mentioned above.

For the weak compactness theorem A we will essentially follow Uhlenbeck's original proof. For the strong compactness theorem E, however, we use an alternative approach by Salamon. This reduces the strong compactness to the weak compactness with the help of a subtle local slice theorem F. It can be used to put the connections into relative Coulomb gauge with respect to the limit connection that is provided by the weak Uhlenbeck compactness theorem A. Then the strong Uhlenbeck compactness theorem E is a consequence of elliptic estimates for the connections. This approach circumvents a further patching argument. It is also useful for the generalization to manifolds with boundary: In the local slice theorem F we establish the relative Coulomb gauge with a suitable boundary condition. This complements the Yang-Mills equation with boundary condition $*F_A|_{\partial M} = 0$ to an elliptic boundary value problem. Furthermore, this line of argument is also suitable for the study of boundary value problems with nonlocal boundary conditions such as described in [Sa, W].

The 'standard' proof of the strong Uhlenbeck compactness theorem E essentially follows the same line of argument as the proof of the weak Uhlenbeck compactness theorem A: One first finds local Coulomb gauges in which one has convergent subsequences and then obtains global gauges from a patching construction. We slightly simplified the patching construction in the proof of theorem A, and also provide the generalization that allows to use this construction for a proof of theorem E.

The local Uhlenbeck gauges are provided by Uhlenbeck's local gauge theorem B for connections with sufficiently small L^q-energy. Here one can use the conformally invariant energy, that is $q = \frac{n}{2}$ (if we assume $n > 2$). That this energy is locally small is ensured by a global bound on the L^p-energy for $p > \frac{n}{2}$. For the proof of the weak compactness theorem A, it would actually suffice to construct these gauges on Euclidean balls. However, if one wants to obtain the stronger convergence in theorem E, then the local gauges have to augment the Yang-Mills equation to an elliptic boundary value problem. This requires the more general form of theorem B, that constructs the Coulomb gauges with respect to a fixed metric on the manifold.

One of the main motivations for this book was to clarify the proof of this local gauge theorem B. It boils down to solving the boundary value problem posed by the Coulomb gauge for the gauge transformation. Then an a priori estimate provides the further estimates involved in the Uhlenbeck gauge. This a priori estimate is based on the L^p-estimate for the operator $\mathrm{d} \oplus \mathrm{d}^*$ on the space of 1-forms that satisfy the boundary condition from the Coulomb gauge (see theorem D).

Uhlenbeck's approach to solving the boundary value problem for the gauge transformation is to first construct a gauge transformation that solves the boundary condition and then use the implicit function theorem to solve the differential equation with homogenous boundary condition. The solution of the boundary condition requires the seemingly obvious theorem C. It asserts that the space of $W^{1,p}$-functions restricted to the boundary of a compact manifold is identical to the space of normal derivatives of $W^{2,p}$-functions. This was proven in higher generality by Agmon, Douglis, Nirenberg, [ADN], but even in this most basic case the proof requires the explicit solution of the Neumann boundary value problem on the half space with inhomogenous boundary conditions.

In this book we pursue the alternative approach suggested by Uhlenbeck: The boundary value problem for the gauge transformation can be directly solved with the implicit function theorem. This involves inhomogenous boundary conditions, so one has to work with boundary value spaces. Moreover, the surjectivity of the linearized operator requires the existence theorem for the Neumann boundary value problem with inhomogenous boundary conditions on L^p-spaces, which brings us back to the work of Agmon, Douglis, Nirenberg.

The L^p-theory for the Neumann boundary value problem with inhomogenous boundary conditions also enters in the elliptic estimates for the strong Uhlenbeck compactness theorem on manifolds with boundary and on manifolds exhausted by compact sets (which necessarily have nonempty boundaries). Thus it seemed appropriate to include an exposition of the Neumann problem in part I. This covers all the results that are required in this book.

Another result that fits well into this book but has no application within it is the local slice theorem F′ that provides a weak relative Coulomb gauge for L^p-connections. This gauge is used in [W] to generalize the compacntess of the moduli space of flat connections to (weakly) flat L^p-connections. This in turn is needed to deal with Lagrangian boundary conditions for anti-self-dual instantons.

The main results

The weak and strong Uhlenbeck compactness theorems deal with sequences of G-connections for compact Lie groups G. More precisely, let $P \to M$ be a principal G-bundle. Throughout this book the base manifold M is a smooth n-manifold with (possibly empty) boundary. If M is compact we will consider sequences in the Sobolev space $\mathcal{A}^{1,p}(P)$ of $W^{1,p}$-connections on P. The group $\mathcal{G}^{2,p}(P)$ of $W^{2,p}$-gauge transformations acts continuously on $\mathcal{A}^{1,p}(P)$. In the case of a noncompact base manifold M we consider the space $\mathcal{A}^{1,p}_{loc}(P)$ with the action of the gauge group $\mathcal{G}^{2,p}_{loc}(P)$.

These Sobolev spaces and actions are carefully defined in the appendices A and B; they are well defined for $p > \frac{n}{2}$. The action of the gauge group is in fact smooth, which leads to a Banach manifold structure of the moduli space $W^{1,p}(P)/\mathcal{G}^{2,p}(P)$ (away from the singularities which actually turn this into an orbifold). This becomes important in the study of moduli spaces of Yang-Mills connections, e.g. in Donaldson theory and Floer homology. However, for the compactness results which we fokus on here, it is enough to know that the gauge action is continuous.

The weak Uhlenbeck compactness theorem (for compact base manifolds) asserts that every subset of the quotient $\mathcal{A}^{1,p}(P)/\mathcal{G}^{2,p}(P)$ that satisfies an L^p-bound on the curvature is weakly compact. This theorem holds for compact manifolds with boundary as well as for closed manifolds.

Theorem A (Weak Uhlenbeck Compactness)
Assume M is a compact Riemannian n-manifold and let $1 < p < \infty$ be such that $p > \frac{n}{2}$. Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P)$ be a sequence of connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ such that $u^{\nu\,*}A^\nu$ converges weakly in $\mathcal{A}^{1,p}(P)$.

Here the compactness of M is crucial. There also is a version of weak Uhlenbeck compactness for manifolds $M = \bigcup_{k\in\mathbb{N}} M_k$ that are exhausted by an increasing sequence of compact submanifolds, i.e. each compact submanifold M_k is contained in the interior of the next, M_{k+1}. In order to extend the weak Uhlenbeck compactness theorem to this situation we shall also assume that each submanifold M_k is a deformation retract of M.[2] This includes for example manifolds with finitely many punctures and cylindrical ends. The following theorem gives the precise formulation of weak Uhlenbeck compactness in this situation. It is a slight generalization of theorem A. Its proof uses an extension argument of Donaldson and Kronheimer [DK].

[2] This condition ensures that every gauge transformation on M_k extends to M. It is an open question whether theorem A′ holds for more general manifolds.

Theorem A′
Assume that $M=\bigcup_{k\in\mathbb{N}}M_k$ *is a Riemannian* n*-manifold exhausted by an increasing sequence of compact submanifolds* M_k *that are deformation retracts of* M*. Let* $1<p<\infty$ *be such that* $p>\frac{n}{2}$*. Let* $(A^\nu)_{\nu\in\mathbb{N}}\subset\mathcal{A}^{1,p}_{loc}(P)$ *be a sequence of connections and for all* $k\in\mathbb{N}$ *suppose that* $\|F_{A^\nu}\|_{L^p(M_k)}$ *is uniformly bounded.*

Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$*) and a sequence of gauge transformations* $u^\nu\in\mathcal{G}^{2,p}_{loc}(P)$ *such that* $u^{\nu\,*}A^\nu|_{M_k}$ *converges weakly in* $\mathcal{A}^{1,p}(P|_{M_k})$ *for all* $k\in\mathbb{N}$*.*

The first step towards the proof of these weak compactness results is to establish the existence of a Coulomb type gauge over small trivializing neighbourhoods $U\subset M$. In a fixed trivialization of $P|_U$ connections are represented by elements of $\mathcal{A}^{1,p}(U)$, the $W^{1,p}$-space of 1-forms with values in $\mathfrak{g}$ (the Lie algebra of G). Gauge transformations are represented by elements of $\mathcal{G}^{2,p}(U)$, the $W^{2,p}$-space of G-valued functions. Now for $A\in\mathcal{A}^{1,q}(U)$ the L^q-energy is defined by

$$\mathcal{E}(A):=\int_U |F_A|^q.$$

Theorem B (Uhlenbeck Gauge)
Fix a Riemannian n*-manifold* M*, a compact Lie group* G*, and let* $1<q\le p<\infty$ *such that* $q\ge\frac{n}{2}$ *and* $p>\frac{n}{2}$*. In case* $q<n$ *assume in addition* $p\le\frac{nq}{n-q}$*. Then there exist constants* C_{Uh} *and* $\varepsilon_{Uh}>0$ *such that the following holds:*

Every point in M *has a neighbourhood* $U\subset M$ *with smooth boundary such that for every connection* $A\in\mathcal{A}^{1,p}(U)$ *with* $\mathcal{E}(A)\le\varepsilon_{Uh}$*, there exists a gauge transformation* $u\in\mathcal{G}^{2,p}(U)$ *such that*

(i) $\mathrm{d}^*(u^*A)=0$, *(iii)* $\|u^*A\|_{W^{1,q}}\le C_{Uh}\|F_A\|_q$,

(ii) $*\,(u^*A)|_{\partial U}=0$, *(iv)* $\|u^*A\|_{W^{1,p}}\le C_{Uh}\|F_A\|_p$.

The domains $U\subset M$ here will be diffeomorphic to the n-ball. For the proof of theorem A it would suffice to establish (i) and (ii) with respect to a metric that is pulled back from the Euclidean metric (on a ball in $\mathbb{R}^n$ whose diameter is comparable to the diameter of U in the given metric on M). However, if one wants to use theorem B for the 'standard' proof of the strong Uhlenbeck compactness theorem E below, then it is important to establish the gauge conditions (i) and (ii) with respect to the fixed metric on the manifold.

The proof of theorem B boils down to solving the boundary value problem posed by (i) and (ii) for the gauge transformation. In Uhlenbeck's original proof she first finds a gauge transformation that meets the boundary condition (ii) and then solves the homogeneous boundary value problem for (i). In this book we will solve the inhomogeneous boundary value problem right away using boundary value spaces (as was suggested by Uhlenbeck in [U2]). That way one needs an existence theorem for the Neumann problem with inhomogeneous boundary conditions. This will be provided in the preliminary part I. The relevant estimate was proven in [ADN] in high generality. In our case it suffices to establish the following fact that also implies the existence of a gauge transformation that satisfies (ii). So this is an alternative proof of [U2, Lemma 2.6].

Theorem C (Agmon, Douglis, Nirenberg)
Let M be a compact Riemannian manifold, let $k \in \mathbb{N}_0$, and let $1 < p < \infty$. Then there is a constant C such that for every $f \in W^{1,p}(M)$ there exists a $u \in W^{2,p}(M)$ that satisfies

$$\tfrac{\partial u}{\partial \nu} = f\big|_{\partial M}, \qquad \|u\|_{W^{2,p}} \leq C\|f\|_{W^{1,p}}.$$

This will be proven along the lines of [ADN] using the Calderon-Zygmund inequality for the Poisson kernel. The Calderon-Zygmund inequality also lies at the heart of theorem D below. Assertion b) was stated by Uhlenbeck [U2] for the unit ball and was proven there in the case $p = 2$.

Theorem D
Let M be a compact Riemannian manifold and let $1 < p < \infty$. Then the following holds.

a) There is a constant C such that for all $A \in W^{1,p}(M, \mathrm{T}^*M)$ with $*A|_{\partial M} = 0$

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p + \|A\|_p\big).$$

b) Suppose in addition $H^1(M;\mathbb{R}) = 0$. Then there exists a constant C such that for all $A \in W^{1,p}(M, \mathrm{T}^*M)$ with $*A|_{\partial M} = 0$

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p\big).$$

Note that assertion b) provides an a priori estimate for connections that satisfy the Uhlenbeck gauge conditions (i) and (ii). This estimate will be used to establish (iii) and (iv) and thus to prove theorem B.

The second step in the proof of the weak Uhlenbeck compactness is a patching argument. One has to patch together the local gauge transformations obtained from theorem B to construct a sequence of global gauge transformations. In the case of a compact base manifold one can use the exponential map and cutoff functions in the Lie algebra. Topological obstructions only arise in the continuation of gauge transformations from a compact subset to the whole exhausted manifold. An argument of Donaldson and Kronheimer [DK, Lemma 4.4.5] proves theorem A′, but this requires the restriction to manifolds exhausted by deformation retracts.

The strong Uhlenbeck compactness theorem concerns sequences of connections that satisfy the Yang-Mills equation

$$\begin{cases} \mathrm{d}_A^* F_A = 0, \\ *F_A|_{\partial M} = 0. \end{cases} \tag{YM}$$

Note that our Yang-Mills equation incorporates a boundary condition. This is the natural boundary condition arising from the variational principle for the Yang-Mills functional on manifolds with boundary. Extrema of the Yang-Mills functional

$$\mathcal{YM}(A) = \int_M |F_A|^2$$

solve the Yang-Mills equation in its weak form: For every $\beta \in \Omega^1(P;\mathfrak{g})$ with compact support (but not necessarily $\beta = 0$ on ∂M)

$$\int_M \langle\, F_A \,,\, \mathrm{d}_A \beta \,\rangle = 0. \tag{wYM}$$

The solutions $A \in \mathcal{A}^{1,p}_{loc}(P)$ of this weak equation will be called weak Yang-Mills connections. In order for this equation to make sense the Sobolev exponent p has to be sufficiently large depending on the dimension $\dim M = n$ of the base manifold. [3] In the case of smooth connections this weak Yang-Mills equation (wYM) is equivalent to the boundary value problem (YM), i.e. the strong Yang-Mills equation. If moreover the base manifold M has no boundary then the weak Yang-Mills equation for smooth connections is equivalent to the usual Yang-Mills equation $\mathrm{d}_A^* F_A = 0$ without boundary condition.

The strong Uhlenbeck compactness theorem for G-bundles over manifolds with (possibly empty) boundary uses above definition of weak Yang-Mills connections (including the boundary condition).

Theorem E (Strong Uhlenbeck Compactness)
Assume M is a compact Riemannian n-manifold. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$ and in case $n = 2$ assume $p > \frac{4}{3}$. Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P)$ be a sequence of weak Yang-Mills connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded.

Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ such that $u^{\nu\,*}A^\nu$ converges uniformly with all derivatives to a smooth connection $A \in \mathcal{A}(P)$.

Again, this theorem generalizes to manifolds which can be exhausted by compact deformation retracts. Moreover, one can perturb the Yang-Mills equation by considering a $\mathcal{C}^\infty$-convergent sequence of metrics g_ν and weak Yang-Mills connections A^ν with respect to the metrics g_ν.

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[3] If we assume $1 < p < \infty$ and $p > \frac{n}{2}$, then this is enough for $n = 1$ and $n \geq 3$. Only for $n = 2$ we need the additional condition $p \geq \frac{4}{3}$, see chapter 9.

Theorem E′
Assume $M = \bigcup_{k\in\mathbb{N}} M_k$ is a Riemannian n-manifold exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$ and in case $n = 2$ assume $p > \frac{4}{3}$. Let $(g_\nu)_{\nu\in\mathbb{N}}$ be a sequence of metrics on M that converges uniformly with all derivatives on every compact set. For all $\nu \in \mathbb{N}$ let $A^\nu \in \mathcal{A}^{1,p}_{loc}(P)$ be a weak Yang-Mills connection with respect to g_ν and suppose that for all $k \in \mathbb{N}$

$$\sup_{\nu\in\mathbb{N}} \|F_{A^\nu}\|_{L^p(M_k)} < \infty.$$

Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^{\nu\,*}A^\nu$ converges uniformly with all derivatives on every compact set to a smooth connection $A \in \mathcal{A}(P)$.

The key to the proofs of these two theorems is the existence of a global relative Coulomb gauge ensured by the local slice theorem F below. The weak Uhlenbeck compactness theorem provides a $W^{1,p}$-weakly convergent subsequence and a limit connection. After a common gauge transformation the limit connection is smooth. Now a further subsequence can be put into relative Coulomb gauge with respect to that limit connection. The strong Uhlenbeck compactness then follows from elliptic estimates for the operator

$$\alpha \mapsto \left(\mathrm{d}^*_{\tilde{A}+\alpha} F_{\tilde{A}+\alpha}\,,\, *F_{\tilde{A}+\alpha}\big|_{\partial M}\,,\, \mathrm{d}^*_{\tilde{A}}\alpha\,,\, *\alpha|_{\partial M}\right),$$

where $\tilde{A} \in \mathcal{A}(P)$ is a smooth connection. This approach by Salamon is different from the proofs of Uhlenbeck [U2] and Donaldson-Kronheimer [DK] (whose 'standard' proof will also be explained in chapter 10). It reduces the strong Uhlenbeck compactness theorem E to the weak compactness theorem A without using a further patching argument (in the case of a compact base manifold). The proof of theorem E′ moreover uses the same extension argument as the proof of theorem A′.

Theorem F (Local Slice Theorem)
Let M be a compact Riemannian n-manifold with smooth boundary (that might be empty). Let $1 < p \le q < \infty$ such that

$$p > \tfrac{n}{2} \qquad \textit{and} \qquad \tfrac{1}{n} > \tfrac{1}{q} > \tfrac{1}{p} - \tfrac{1}{n}.$$

Fix a reference connection $\hat{A} \in \mathcal{A}^{1,p}(P)$ and a constant $c_0 > 0$. Then there exist constants $\delta > 0$ and C such that the following holds: For every $A \in \mathcal{A}^{1,p}(P)$ with

$$\|A - \hat{A}\|_q \le \delta \qquad \textit{and} \qquad \|A - \hat{A}\|_{W^{1,p}} \le c_0$$

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(P)$ such that

(i) $\mathrm{d}^*_{\hat{A}}(u^*A - \hat{A}) = 0$, *(iii)* $\|u^*A - \hat{A}\|_q \le C\|A - \hat{A}\|_q$,

(ii) $*(u^*A - \hat{A})|_{\partial M} = 0$, *(iv)* $\|u^*A - \hat{A}\|_{W^{1,p}} \le C\|A - \hat{A}\|_{W^{1,p}}$.

This theorem asserts the existence of a local slice through $\hat{A}$ that is transversal to the orbits of the gauge action. (This is because the infinitesimal action of the gauge group at $\hat{A}$ is given by $\mathrm{d}_{\hat{A}}$.) For every L^q-close connection A one then finds a gauge equivalent connection in the local slice, but can keep control of the $W^{1,p}$-norm. This goes beyond a simple application of the implicit function theorem since it only requires a $W^{1,p}$-bound on A, not $W^{1,p}$-closeness to $\hat{A}$.

There also is an L^p-version of the local slice theorem that will be proven in this book. In order to state the weak Coulomb equation involved we use the notation $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ for the associated bundle arising from the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Then the difference of any two smooth connections is an element of $\Gamma(\mathrm{T}^*M \otimes \mathfrak{g}_P)$.

Theorem F′ ($\mathbf{L^p}$-Local Slice Theorem)
Let M be a compact Riemannian n-manifold with smooth, possibly empty boundary. Let $2 \leq p < \infty$ be such that $p > n$ and fix a reference connection $\hat{A} \in \mathcal{A}^{0,p}(P)$. Then there exist constants $\delta > 0$ and C such that the following holds.

For every $A \in \mathcal{A}^{0,p}(P)$ with $\|A - \hat{A}\|_p \leq \delta$ there exists a gauge transformation $u \in \mathcal{G}^{1,p}(P)$ such that

$$\int_M \langle u^*A - \hat{A}\,,\, \mathrm{d}_{\hat{A}}\eta \rangle = 0 \qquad \forall \eta \in \Gamma(\mathfrak{g}_P)$$

$$\text{and} \qquad \|u^*A - \hat{A}\|_p \leq C\|A - \hat{A}\|_p .$$

Following [CGMS] both local slice theorems F and F′ will be proven by Newton's iteration method. In fact, theorem F′ could also be proven by the implicit function theorem and one extra estimate. However, we use this easier case to illustrate the iteration method. For theorem F this iteration is considerably more complicated due to the boundary terms and the fact that the $W^{1,p}$-norm of the connection is only assumed to be bounded, not small. This, however, is just the setting that is obtained from the weak Uhlenbeck compactness theorem A: The connections converge in the weak $W^{1,p}$-topology, so they are $W^{1,p}$-bounded and they converge strongly only with respect to an L^q-norm.

Theorem F′ is used in [W] to generalize the regularity of flat connections to L^p-connections: A connection of class $W^{1,p}$ is called flat if its curvature vanishes, which is a partial differential equation of first order. One can use theorem F to prove that every flat connection is gauge equivalent to a smooth connection. For connections of class L^p one can introduce the notion of 'weak flatness' by a weak equation. The combination of the weak flatness and the weak relative Coulomb gauge provided by theorem F′ then constitutes an elliptic system whose L^p-solutions are in fact smooth. This proves that every weakly flat connection is gauge equivalent to a smooth connection.

Outline

This book is organized in four parts. Part I is of preliminary nature. It gives an exposition of the Neumann problem in chapters 1 to 3. We give proofs of a number of results that are wellknown but for which there seem to be no explicit proofs in the standard textbooks. For example, the L^2-regularity theorem 1.3 for weak solutions of the Neumann problem requires no minimum regularity of the solution. We also prove the regularity and existence results for L^p-spaces with general $p > 1$. Moreover, theorem C is of central importance for the Neumann problem with inhomogeneous boundary conditions, but some textbooks simply omit it. Here we give a proof that uses the methods of [ADN] but is considerably easier than their treatment of general boundary value problems, see theorem 3.4.

In chapter 4 some results on the Neumann problem are generalized to sections of vector bundles with nonsmooth connections. These are used in the Newton iteration for the local slice theorems F and F′.

In part II we prove the weak Uhlenbeck compactness theorems A and A′, see theorems 7.1 and 7.5. Firstly, chapter 5 provides regularity results for 1-forms which correspond to Hodge theory on manifolds with boundary. These are used to prove the L^p-estimates of theorem D for the operator $\mathrm{d} \oplus \mathrm{d}^*$, restated in theorem 5.1. Moreover, the regularity theory for Yang-Mills connections will again make use of these results.

In chapter 6 we then prove the Uhlenbeck local gauge theorem B, restated as theorem 6.1. We use boundary value spaces instead of Uhlenbeck's explicit construction of a gauge transformation that meets the boundary condition. We also filled in a lot of technicalities: Uhlenbeck proves the theorem for one model domain, that is the Euclidean unit ball [U2, Thm.2.1]. We show in theorem 6.3 that the theorem on the unit ball in fact holds for all metrics that are sufficiently $\mathcal{C}^2$-close to the Euclidean metric. Moreover, in order to generalize the theorem to manifolds with boundary and a fixed metric, we prove the same result for a second model domain, the "egg squeezed to the boundary". We furthermore explain the rescaling trick that proves the existence of the Uhlenbeck gauge in sufficiently small neighbourhoods on general manifolds.

Chapter 7 provides the patching constructions that complete the proof of the weak Uhlenbeck compactness theorems. In the case of a compact base manifold we have slightly modified Uhlenbeck's patching argument. The basic result is that every set of transition functions posesses a $\mathcal{C}^0$-neighbourhood of sets of transition functions that all define the same bundle. In [U2, Prop.3.2] the underlying cover has to be finite and the radius of the $\mathcal{C}^0$-neighbourhood depends on the number of covering patches and the set of transition functions. Our patching lemma 7.2 works for all countable covers of manifolds and the radius of the $\mathcal{C}^0$-neighbourhood simply is the radius of a convex geodesic ball in the Lie group. The proof of theorem A (restated as theorem 7.1) is based on this patching lemma.

The generalization of weak Uhlenbeck compactness to noncompact manifolds uses the extension argument of Donaldson and Kronheimer [DK, Lemma 4.4.5].

We explain that argument in lemma 7.8 and use it to prove proposition 7.6, which is a general tool for extending compactness results for moduli spaces over compact manifolds to base manifolds that are exhausted by compact deformation retracts. This result is then used to prove theorem A′, restated as theorem 7.5. It can again be used for the proof of theorem E′. It should be stressed that the extension argument requires that every gauge transformation on one of the exhausting compact submanifolds extends to the whole manifold. This is ensured by our assumption that the exhausting submanifolds are deformation retracts of the manifold.

Part III concerns the strong Uhlenbeck compactness theorems E and E′ (theorems 10.1 and 10.3). Here the generalization to manifolds with boundary requires to supplement the Yang-Mills equation with a boundary condition. This boundary value problem also occurs in the generalization to manifolds that are exhausted by compact sets, since these compact submanifolds necessarily have boundaries.

In chapter 8 we give proofs of the local slice theorem F and its L^p-version, theorem F′ (see theorem 8.1 and 8.3). These are adaptations of the Newton iteration method used for [CGMS, Thm.A.1] to manifolds with boundary and to L^p-connections. Moreover, we construct the relative Coulomb gauge with respect to different metrics under the same assumptions on the connection. This is needed for the case of varying metrics in theorem E′.

Chapter 9 introduces the Yang-Mills equation with boundary condition. We prove the smoothness of Yang-Mills connections up to a gauge transformation both on compact manifolds and on manifolds that are exhausted by compact deformation retracts, see theorem 9.4. This is done by the iteration of two regularity results. These already include estimates for the proof of the strong Uhlenbeck compactness.

Chapter 10 is devoted to the proofs of theorem E (theorem 10.1) and theorem E′ (theorem 10.3). Unlike to the 'standard' proof (which we also explain) this approach by Salamon requires no further patching construction for the proof of theorem E due to the use of the global relative Coulomb gauge provided by the local slice theorem F. For theorem E′ one again uses proposition 7.6, which relies on the extension argument of Donaldson and Kronheimer.

These last two chapters and chapter 5 contain careful details of the bootstrapping analysis. These are standard procedures but they did not seem entirely obvious, especially not for manifolds with boundary, and they have to be adapted separately to the case of noncompact manifolds exhausted by compact sets. Moreover, in order to obtain the compactness results for varying metrics, one has to obtain uniform constants for a small neighbourhood of metrics in all these estimates. This does not require a lot of extra work, but one always has to take care of what the constants depend on.

Part IV consists of a number of appendices that are designed to make this book as selfcontained as possible. Appendix A gives a brief introduction to gauge theory. This is not meant as an exposition but rather sets up the notation that is used throughout the book. Moreover, we prove some fundamental estimates on the energy and gauge transformations which will be needed frequently.

In appendix B one can find the definition of Sobolev spaces of sections of vector bundles and – more generally – fibre bundles. This leads to the definition of the Sobolev spaces of connections and gauge transformations. We also state all Sobolev embedding results that will be needed in the book and we give a coordinate free proof of a trace theorem (concerning the restriction of Sobolev functions to the boundary of a compact manifold).

Theorem C.2 in appendix C states a criterion for L^p-multipliers due to Mihlin [M]. It is much easier to check than the usual criteria e.g. in [St1]. We show how the criterion of Mihlin implies the standard criterion. Then we use this criterion to prove the Calderon-Zygmund inequality (theorem C.3). Furthermore, we use techniques of [ADN] to give a proof of the estimates on the Poisson kernel (theorem C.4) that are needed for theorem C. Similar techniques are used to prove a version of the mollifier theorem C.5 that again is more general than the one in most standard textbooks: We do not require compact support of the functions that are proven to converge to the Dirac distribution.

Appendix D states (without proofs) the main results on the Dirichlet problem. Appendix E states some results from Functional Analysis that are used at crucial places in this book.

Part I

The Neumann Problem

Throughout this part let M be a compact oriented Riemannian manifold with (possibly empty) boundary ∂M. We shall denote by $\Delta = \mathrm{d}^*\mathrm{d}$ the Hodge Laplacian on functions and by ν the outward unit normal vector field to ∂M. This part deals with the **Neumann boundary value problem**

$$\begin{cases} \Delta u = f & \text{on } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{NP}$$

Let $\mathcal{D}(M)$ be the space of continuous linear functionals on $\mathcal{C}^\infty(M)$ and

$$\mathcal{C}^\infty_\nu(M) := \left\{ \psi \in \mathcal{C}^\infty(M) \,\middle|\, \tfrac{\partial \psi}{\partial \nu} = 0 \right\}.$$

We denote by $\langle\, u\,,\, \phi\,\rangle$ the pairing of a distribution $u \in \mathcal{D}(M)$ with $\psi \in \mathcal{C}^\infty(M)$. A distribution $u \in \mathcal{D}(M)$ is called a **weak solution** of (NP) if

$$\langle\, u\,,\, \Delta\psi\,\rangle = \langle\, f\,,\, \psi\,\rangle \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M). \tag{wNP}$$

Lemma N *Let $p \geq 1$, $f \in \mathcal{D}(M)$, and $u \in W^{2,p}(M)$. Then u satisfies* (NP) *if and only if it satisfies* (wNP)*. If this is the case then $f \in L^p(M)$.*

Proof: Choose a sequence $u_i \in \mathcal{C}^\infty(M)$ that converges to u in the $W^{2,p}$-norm. Then $\frac{\partial u_i}{\partial \nu}$ converges to $\frac{\partial u}{\partial \nu}$ in $L^1(\partial M)$ by theorem B.10. Moreover, lemma 5.6 (i) asserts that $*\mathrm{d}v|_{\partial M} = \frac{\partial v}{\partial \nu}\,\mathrm{dvol}_{\partial M}$ holds for all functions v. Thus for every $\psi \in \mathcal{C}^\infty_\nu$ Stokes' theorem gives

$$\begin{aligned} \int_M u\,\Delta\psi\,\mathrm{dvol}_M &= -\lim_{i\to\infty}\int_M u_i\,\mathrm{d}*\mathrm{d}\psi \\ &= \lim_{i\to\infty}\left(-\int_M \psi\,\mathrm{d}*\mathrm{d}u_i + \int_M \mathrm{d}(\psi*\mathrm{d}u_i - u_i*\mathrm{d}\psi)\right) \\ &= \lim_{i\to\infty}\left(\int_M \psi\,\Delta u_i\,\mathrm{dvol}_M + \int_{\partial M}\left(\psi\,\tfrac{\partial u_i}{\partial \nu} - u_i\,\tfrac{\partial \psi}{\partial \nu}\right)\mathrm{dvol}_{\partial M}\right) \\ &= \int_M \psi\,\Delta u\,\mathrm{dvol}_M + \int_{\partial M}\psi\,\tfrac{\partial u}{\partial \nu}\,\mathrm{dvol}_{\partial M}. \end{aligned}$$

This immediately implies (wNP) if u satisfies (NP). Now assume (wNP), then the previous calculation implies

$$\int_M \psi\,\Delta u + \int_{\partial M}\tfrac{\partial u}{\partial \nu}\,\psi = \langle\, f\,,\, \psi\,\rangle \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M).$$

Testing with $\psi \in \mathcal{C}^\infty(M)$ that have compact support in the interior of M we obtain $f = \Delta u \in L^p(M)$. Hence $\int_{\partial M}\frac{\partial u}{\partial \nu}\,\psi = 0$ for all $\psi \in \mathcal{C}^\infty_\nu(M)$. This implies $\frac{\partial u}{\partial \nu} = 0$ since every smooth function on ∂M can be smoothly extended to M such that the normal derivative at the boundary vanishes. □

This part is organized as follows: We start with an exposition of the L^2-theory for the Neumann problem with homogeneous boundary condition. The subsequent chapter 2 shows how these results extend to L^p-spaces for $1 < p < \infty$. Then, in chapter 3, we establish the required results for the Neumann problem with inhomogeneous boundary conditions.

All these results are formulated for real-valued functions, but they generalize directly to functions with values in a Hilbert space. One just has to replace the multiplication of two functions by the pointwise inner product on the Hilbert space. However, there is no natural extension of this regularity theory to functions with values in a Banach space – the complete metric on the target space is crucial.

In chapter 4 we generalize some of these results to a Neumann type boundary value problem for sections of vector bundles. This will be needed for the local slice theorem in chapter 8.

Chapter 1

L^2-Theory

The regularity theory makes use of the generalization of $W^{k,2}$-Sobolev spaces to $k \leq 0$. We denote $W^{0,2}(M) := L^2(M)$.

Definition 1.1 *Let $k \in \mathbb{N}_0$. Then $W^{-k,2}(M) \subset \mathcal{D}(M)$ is defined as the space of distributions $u \in \mathcal{D}(M)$ for which there exists a constant C such that*

$$|\langle u\,,\,\psi\rangle| \leq C\|\psi\|_{W^{k,2}} \qquad \forall \psi \in \mathcal{C}^\infty(M). \tag{1.1}$$

The natural norm on this space is the operator norm $\|\cdot\|_{W^{-k,2}}$, i.e. the smallest constant C such that (1.1) holds.

Remark 1.2

(i) $W^{-k,2}(M)$ can be naturally identified with the dual space of $W^{k,2}(M)$:

On the one hand, every continuous linear functional on $W^{k,2}(M)$ also belongs to $W^{-k,2}(M)$ since $\mathcal{C}^\infty(M) \subset W^{k,2}(M)$. Conversely, every distribution that lies in $W^{-k,2}(M)$ uniquely extends to a continuous linear functional on $W^{k,2}(M)$ since $\mathcal{C}^\infty(M)$ is dense in $W^{k,2}(M)$.

We extend the notation $\langle\cdot,\cdot\rangle$ to the pairing of $W^{-k,2}(M)$ with $W^{k,2}(M)$.

(ii) For $k = 0$ the spaces $W^{-0,2}(M)$ and $W^{0,2}(M) = L^2(M)$ are identified by the Riesz representation theorem. In this case $\langle\cdot,\cdot\rangle$ is the L^2-inner product.

(iii) The Sobolev embedding theorem implies

$$\mathcal{C}^\infty(M) = \bigcap_{k\in\mathbb{N}} W^{k,2}(M).$$

The space of distributions can thus be represented by

$$\mathcal{D}(M) = \bigcup_{k\in\mathbb{N}} W^{-k,2}(M).$$

Assume otherwise that there exist $u \in \mathcal{D}(M)$ and $\phi_k \in \mathcal{C}^\infty(M)$ such that $|\langle u, \phi_k \rangle| \geq k\|\phi_k\|_{W^{k,2}}$ and $\|\phi_k\|_{W^{k,2}} = 1$ for all $k \in \mathbb{N}$. Then the sequence $\frac{1}{k}\phi_k$ converges to 0 in $\mathcal{C}^\infty(M)$ since $\|\frac{1}{k}\phi_k\|_{W^{j,2}} \leq \frac{1}{k}$ for all $k \geq j$. But $|\langle u, \frac{1}{k}\phi_k\rangle| \geq 1$ for all $k \in \mathbb{N}$ hence u is not a continuous functional on $\mathcal{C}^\infty(M)$.

In this chapter we give proofs of the following L^2-estimate, regularity, and existence results for the Neumann boundary value problem:

Theorem 1.3 *Let $k \in \mathbb{Z}$ and suppose that $u \in \mathcal{D}(M)$ is a weak solution of the Neumann problem* (wNP) *with $f \in W^{k,2}(M)$. Then $u \in W^{k+2,2}(M)$.*

Theorem 1.4 *Let $k \in \mathbb{N}_0$. Then there exist constants C, C' such that for all $u \in W^{k+2,2}(M)$*

$$\|u\|_{W^{k+2,2}} \leq C\big(\|\Delta u\|_{W^{k,2}} + \|u\|_{W^{k,2}}\big) \qquad \textit{if} \quad \tfrac{\partial u}{\partial \nu} = 0,$$
$$\|u\|_{W^{k+2,2}} \leq C'\|\Delta u\|_{W^{k,2}} \qquad \textit{if} \quad \tfrac{\partial u}{\partial \nu} = 0 \quad \textit{and} \quad \textstyle\int_M u = 0.$$

Theorem 1.5 *Let $f \in W^{k,2}(M)$ for some $k \in \mathbb{N}_0$. Then there exists a solution $u \in W^{k+2,2}(M)$ of* (NP) *if and only if $\int_M f = 0$. This solution is unique up to an additive constant.*

In most textbooks theorem 1.3 can only be found under the assumption of some minimum regularity of u, e.g. in [T, ch.5,Prop.7.4] this is proven under the assumptions $k \geq 0$ and $u \in W^{1,2}(M)$.

In order to prove theorems 1.3 and 1.4 we first consider the operator $\Delta + 1$. It is represented by the following bilinear form on $W^{1,2}(M)$: For all $u, v \in W^{1,2}(M)$

$$A(u,v) := \int_M \left(\mathrm{d}u \wedge *\mathrm{d}v + uv\,\mathrm{dvol}_M\right).$$

For $\psi \in \mathcal{C}^\infty_\nu(M)$ Stokes' theorem gives

$$\begin{aligned} A(u,\psi) &= \int_M \mathrm{d}u \wedge *\mathrm{d}\psi \;+\; \langle u, \psi \rangle \\ &= \int_{\partial M} u\tfrac{\partial \psi}{\partial \nu}\,\mathrm{dvol}_{\partial M} - \int_M u\,\mathrm{d} * \mathrm{d}\psi \;+\; \langle u, \psi \rangle \\ &= \langle u, \Delta\psi + \psi \rangle . \end{aligned}$$

From this identity and lemma N we draw the following conclusion.

Remark 1.6

(i) Let $f \in W^{-1,2}(M)$ and $u \in W^{1,2}(M)$ satisfy

$$A(u, \psi) = \langle f , \psi \rangle \qquad \forall \psi \in W^{1,2}(M). \tag{1.2}$$

Then u is a weak solution of the Neumann problem (wNP) where f is replaced by $f - u$.

(ii) Let $f \in W^{-1,2}(M)$ and $u \in W^{2,2}(M)$. Then (1.2) holds if and only if u is a strong solution of

$$\begin{cases} \Delta u + u = f & \text{on } M, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial M. \end{cases} \tag{1.3}$$

If this is the case then $f \in L^2(M)$.

The ellipticity of $\Delta + 1$ is reflected in the fact that the bilinear form A is coercive. A first consequence is the following lemma.

Lemma 1.7 *For every $f \in W^{-1,2}(M)$ there is a unique solution $u \in W^{1,2}(M)$ of (1.2). This moreover satisfies*

$$\|u\|_{W^{1,2}} \le \|f\|_{W^{-1,2}}.$$

Proof: The bilinear form A is bounded and coercive: For all $u, v \in W^{1,2}(M)$

$$|A(u,v)| \le \|u\|_{W^{1,2}} \|v\|_{W^{1,2}} \qquad \text{and} \qquad A(u,u) \ge \|u\|^2_{W^{1,2}}.$$

Thus the Lax-Milgram theorem E.2 asserts that for every $f \in W^{-1,2}(M)$ there exists a unique $u \in W^{1,2}(M)$ that solves (1.2). The estimate follows from

$$\|u\|^2_{W^{1,2}} \;\le\; A(u,u) \;=\; \langle f , u \rangle \;\le\; \|f\|_{W^{-1,2}} \|u\|_{W^{1,2}}.$$

□

It is then a matter of the regularity of u to prove that $\Delta + 1$ is a bijection between $W^{k,2}(M)$ and $W^{k+2,2}(M)$.

Theorem 1.8 *Let $k \ge -1$ be an integer. Then for every $f \in W^{k,2}(M)$ there exists a unique solution $u \in W^{k+2,2}(M)$ of (1.2). Moreover, there exists a constant C such that*

$$\|u\|_{W^{k+2,2}} \le C \|f\|_{W^{k,2}}. \tag{1.4}$$

Proof: Lemma 1.7 proves the theorem in case $k = -1$. In case $k \ge 0$ it remains to prove that the unique $u \in W^{1,2}(M)$ from the lemma actually lies in $W^{k+2,2}(M)$ and satisfies (1.4). This regularity will be proven by induction on k. An intermediate step is the following:

<u>Claim:</u> *Let X be a smooth vector field on M that is tangential to ∂M and whose flow ϕ_t preserves the unit normal, $\mathrm{d}\phi_t(\nu)=\nu$. Then for every $k\in\mathbb{N}_0$ there is a constant C such that the following holds: Let $f\in W^{k,2}(M)$ and $u\in W^{k+1,2}(M)$ satisfy (1.2). Then $\mathcal{L}_X u\in W^{k+1,2}(M)$ and*

$$\|\mathcal{L}_X u\|_{W^{k+1,2}} \leq C\bigl(\|f\|_{W^{k,2}} + \|u\|_{W^{k+1,2}}\bigr). \tag{1.5}$$

We will first prove the claim and the theorem for $k=0$. Then as induction step we assume the theorem to hold for some $k\geq 0$, prove the claim for $k+1$, and then deduce the theorem for $k+1$.

<u>Proof of the claim for $k=0$:</u>
The time-t-flow ϕ_t of the vector field X is a diffeomorphism of M since X is tangential to the boundary. Thus one can consider

$$D_t u \;:=\; \tfrac{1}{t}\,(\phi_t^* u - u)\;\in W^{1,2}(M).$$

When $t\to 0$ then $D_t u$ converges to $\mathcal{L}_X u$ in the L^2-norm. Now indicating the dependence on a metric by subscripts we calculate for all $\psi\in W^{1,2}(M)$

$$\begin{aligned}
&A(D_t u\,,\,\psi)\\
&= \int_M \mathrm{d}\psi\wedge *_g \mathrm{d}(D_t u) + \int_M \psi\cdot D_t u\cdot \mathrm{dvol}_M\\
&= \tfrac{1}{t}\left[\int_M \phi_t^*\bigl(\mathrm{d}\phi_{-t}^*\psi\wedge *_{\phi_{-t}^* g}\mathrm{d}u\bigr) - \int_M \mathrm{d}\psi\wedge *_g \mathrm{d}u\right]\\
&\quad + \tfrac{1}{t}\left[\int_M \phi_t^*\,\bigl(\phi_{-t}^*\psi\cdot u\cdot \phi_{-t}^*\,\mathrm{dvol}_M\bigr) - \int_M \psi\cdot u\cdot\,\mathrm{dvol}_M\right]\\
&= \int_M \tfrac{1}{t}\mathrm{d}\bigl(\phi_{-t}^*\psi-\psi\bigr)\wedge *_g\mathrm{d}u + \int_M \mathrm{d}\phi_{-t}^*\psi\wedge\tfrac{1}{t}\bigl(*_{\phi_{-t}^* g} - *_g\bigr)\mathrm{d}u\\
&\quad + \int_M \tfrac{1}{t}\bigl(\phi_{-t}^*\psi-\psi\bigr)\cdot u\cdot\,\mathrm{dvol}_M + \int_M \phi_{-t}^*\psi\cdot u\cdot\tfrac{1}{t}\bigl(\phi_{-t}^*\,\mathrm{dvol}_M - \,\mathrm{dvol}_M\bigr)\\
&= -A(u, D_{-t}\psi) + \int_M \mathrm{d}\psi\wedge\tfrac{1}{t}(*_g - *_{\phi_t^* g})\mathrm{d}\phi_t^* u\\
&\quad + \int_M \psi\cdot\phi_t^* u\cdot\tfrac{1}{t}\bigl(\,\mathrm{dvol}_M - \phi_t^*\,\mathrm{dvol}_M\bigr)\\
&= -\,\langle\, f\,,\, D_{-t}\psi\,\rangle - \int_M \mathrm{d}\psi\wedge \mathrm{HS}_t(\mathrm{d}\phi_t^* u) - \int_M \psi\cdot\phi_t^* u\cdot D_t\,\mathrm{dvol}_M .
\end{aligned}\tag{1.6}$$

Here we denote by $\mathrm{HS}_t := \frac{1}{t}(*_{\phi_t^* g} - *_g)$ a pointwise operator on 1-forms. When $t\to 0$ this operator uniformly converges to the continuous operator HS_0 defined as follows: For every 1-form $A=\sum_{j=1}^n A_j \mathrm{d}x^j$

$$\mathrm{HS}_0(A) = \sum_{i,j=1}^n \tfrac{\partial}{\partial t}\Bigl(\sqrt{\det(\phi_{-t}^* g)}(\phi_{-t}^* g)^{ij}\Bigr)\Big|_{t=0}(-1)^{i-1}A_j \mathrm{d}x^1\wedge\ldots\overset{\vee}{\mathrm{d}x^i}\ldots\wedge\mathrm{d}x^n .$$

Moreover, we have extended the notation D_t to denote the difference quotient that converges to the Lie derivative of the volume form. So if $D_t\,\mathrm{dvol}_M = \Delta_t \cdot \mathrm{dvol}_M$ then the functions Δ_t uniformly converge to $\mathrm{div}X$ as $t \to 0$. Now we find constants $\delta > 0$ (depending on u) and C (independent of u) such that for all $t \in (0, \delta)$ and $\psi \in W^{1,2}(M)$

$$\begin{aligned} \big|A(D_t u\,,\,\psi)\big| &\leq \|f\|_2 \|D_{-t}\psi\|_2 + \|\mathrm{d}\psi\|_2 \|\mathrm{HS}_t\|_\infty \|\mathrm{d}\phi_t^* u\|_2 + \|\psi\|_2 \|\phi_t^* u\|_2 \|\Delta_t\|_\infty \\ &\leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big)\|\psi\|_{W^{1,2}}. \end{aligned} \tag{1.7}$$

Firstly, we used that ϕ_t depends smoothly on t, so $\phi_t^* u$ converges to u in the $W^{1,2}$-norm as $t \to 0$. Thus one has $\|\phi_t^* u\|_2 \leq 2\|u\|_2$ and $\|\mathrm{d}\phi_t^* u\|_2 \leq 2\|u\|_{W^{1,2}}$ for sufficiently small $t > 0$. (If $\|u\|_2 = 0$ then $u = 0$ and hence $\phi_t^* u \equiv 0$ for all $t > 0$.) The uniform convergence of the functions Δ_t and the operators HS_t provides further constants, so it remains to consider the term $\|D_{-t}\psi\|_2$. For this purpose first assume that ψ is smooth, then for all $t > 0$ and $x \in M$

$$\begin{aligned} \big|\phi_{-t}^*\psi(x) - \psi(x)\big| &= \left|\int_0^t \tfrac{\partial}{\partial s}\big(\phi_{-s}^*\psi(x)\big)\,\mathrm{d}s\right| \\ &\leq \int_0^t \big|\mathrm{d}_{\phi_{-s}(x)}\psi(X(\phi_{-s}(x))\big|\,\mathrm{d}s \\ &\leq \|X\|_\infty t^{\frac{1}{2}} \left(\int_0^t \big|\mathrm{d}_{\phi_{-s}(x)}\psi\big|^2 \mathrm{d}s\right)^{\frac{1}{2}}. \end{aligned}$$

Note that ϕ_{-s} is a diffeomorphism of M for all $s \in \mathbb{R}$, thus there are nonvanishing functions g_s such that $\phi_{-s}^*\,\mathrm{dvol}_M = g_s \cdot \mathrm{dvol}_M$. Here $g_0 \equiv 1$ and thus $\|g_s^{-1}\|_\infty \leq 2$ for all $s \in [0, \delta)$ if we make a sufficiently small choice of $\delta > 0$. Now use the previous inequality to obtain for all $t \in (0, \delta)$

$$\begin{aligned} \big\|D_{-t}\psi\big\|_2^2 &= \int_M t^{-2}\big|\phi_{-t}^*\psi - \psi\big|^2\,\mathrm{dvol}_M \\ &\leq \|X\|_\infty^2 t^{-1} \int_M \int_0^t |\mathrm{d}\psi|^2 \circ \phi_{-s}\,\mathrm{d}s\,\mathrm{dvol}_M \\ &\leq \|X\|_\infty^2 t^{-1} \int_0^t \int_M |\mathrm{d}\psi|^2 \circ \phi_{-s} \cdot 2\phi_{-s}^*\mathrm{dvol}_M\,\mathrm{d}s \\ &= 2\|X\|_\infty^2 \|\mathrm{d}\psi\|_2^2. \end{aligned}$$

So we find a constant C such that $\|D_{-t}\psi\|_2 \leq C\|\psi\|_{W^{1,2}}$ holds for all smooth ψ and sufficiently small $t > 0$ (not depending on ψ), and this inequality extends to all $\psi \in W^{1,2}(M)$. Thus we have established (1.7).

For all $t \in (0, \delta)$ this shows that $A(D_t u\,, \cdot)$ is a continuous linear functional on $W^{1,2}(M)$ and thus equals $\langle\, F_t\,, \cdot\,\rangle$ for some $F_t \in W^{-1,2}$ with

$$\|F_t\|_{W^{-1,2}} \leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big).$$

Here C is a t-independent constant and now the theorem for the case $k = -1$ (lemma 1.7) asserts that for all $t \in (0, \delta)$

$$\|D_t u\|_{W^{1,2}} \leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big).$$

Finally, by the Banach-Alaoglu theorem B.4 there exists a sequence $t_i \to 0$ such that $D_{t_i} u$ weakly converges in $W^{1,2}(M)$. The limit has to be $\mathcal{L}_X u$ since the sequence already converges L^2-strongly to this function. This proves that $\mathcal{L}_X u \in W^{1,2}(M)$, and due to the lower semicontinuity of the norm with respect to weak convergence the estimate carries over to the limit,

$$\|\mathcal{L}_X u\|_{W^{1,2}} \leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big).$$

Proof of the theorem for $k = 0$:
Let $f \in L^2(M)$, then lemma 1.7 provides a unique solution $u \in W^{1,2}(M)$ of (1.2) and we have to prove that in fact $u \in W^{2,2}(M)$. Moreover, in order to obtain the estimate on u it suffices to find a constant C such that

$$\|u\|_{W^{2,2}} \leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big).$$

Then estimate (1.4) follows from lemma E.3 due to the uniqueness of u (i.e. injectivity of the operator $D : u \mapsto f$) and the compactness of the Sobolev embedding $W^{2,2}(M) \hookrightarrow W^{1,2}(M)$.

Choose a finite atlas $M = \bigcup_{i=1}^N U_i$ with charts $\Phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n$. Let x^i and ∂_i denote the coordinate functions and vector fields respectively. The coordinate charts that intersect the boundary can be chosen such that ∂M is mapped to the hyperplane $\{x^1 = 0\}$ and $\Phi_i^* \partial_1|_{\partial M}$ is the outward unit normal ν. In order to achieve this one extends ν to a vector field on M. Then the flow of this vector field defines a tubular neighbourhood diffeomorphism $(-\varepsilon, 0] \times \partial M \to M$. Its first component can be combined with a chart of ∂M to give the required chart of M.

Now by remark B.1 it suffices to prove that $\partial_j \partial_\ell (u \circ \Phi_i^{-1})$ lies in $L^2(V_i)$ for all $i = 1, \dots N$ and $j, \ell = 1, \dots n$ and that for some finite constant C

$$\|\partial_j \partial_\ell (u \circ \Phi_i^{-1})\|_2 \leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big). \tag{1.8}$$

Unless U_i intersects the boundary and $j = \ell = 1$ this follows from the already proven claim (for $k = 0$). For that purpose consider a smooth vector field X on M that equals $\Phi_i^* \partial_j$ on U_i and is cut off as follows:

If $U_i \cap \partial M = \varnothing$ then X is cut off such that it vanishes near ∂M and hence satisfies the requirements of the claim.

If U_i intersects ∂M but $j \neq 1$ then X is automatically tangential to $\partial M \cap U_i$. Moreover, both X and ν are coordinate vector fields on $\partial M \cap U_i$, thus their flows commute. Now X is cut off such that still $\mathrm{d}\phi_t(\nu) = \nu$ holds on ∂M. For this purpose let U_i lie within a slightly larger chart $\tilde{\Phi}_i : \tilde{U}_i \to (-\tilde{\varepsilon}, 0] \times \tilde{W}$, where

$\overline{U}_i \subset \tilde{U}_i \subset M$, $0 < \varepsilon < \tilde{\varepsilon}$, and W with $\overline{W} \subset \tilde{W} \subset \mathbb{R}^{n-1}$ is the chart of ∂M. In this chart of M a suitable cutoff function for $\tilde{\Phi}_i^* \partial_j$ is $f \cdot g$, where $f = f(x^2, \ldots, x^n) \equiv 1$ on W and vanishes near $\partial\tilde{W}$, and $g = g(x^1) \equiv 1$ on $(-\varepsilon, 0]$ and vanishes near $-\tilde{\varepsilon}$. Then X and its flow ϕ_t are independent of the x^1-coordinate in a neighbourhood of the boundary $\{x^1 = 0\}$, and hence $\mathrm{d}\phi_t(\nu) = \nu$ holds on ∂M.

Since the vector field X satisfies the requirements of the claim we obtain $\mathcal{L}_X u \in W^{1,2}(M)$ and $\|\mathcal{L}_X u\|_{W^{1,2}} \leq C(\|f\|_2 + \|u\|_{W^{1,2}})$ for some constant C. Thus $\partial_j \partial_\ell (u \circ \Phi_i^{-1}) = \partial_\ell(\mathcal{L}_X u \circ \Phi_i^{-1})$ lies in $L^2(V_i)$ and (1.8) holds.

It remains to prove (1.8) for charts at the boundary and $j = \ell = 1$. So let the chart U_i intersect the boundary, then we have to consider the function $v := u \circ \Phi_i^{-1}$ on $V := \Phi_i(U_i) \subset \mathbb{R}^n$. Let ψ be any smooth function with compact support in the interior of V. Denote by e the unit vector in x^1-direction, then for sufficiently small t also $te + \operatorname{supp}\psi$ lies in V and we obtain

$$
\begin{aligned}
&\int_V \tfrac{1}{t}\big(\partial_1 v(x+te) - \partial_1 v(x)\big)\psi(x)\,\mathrm{d}^n x \\
&= \tfrac{1}{t}\left(\int_{te+\operatorname{supp}\psi} \partial_1 v(y)\,\psi(y-te)\,\mathrm{d}^n y - \int_{\operatorname{supp}\psi} \partial_1 v(x)\,\psi(x)\,\mathrm{d}^n x\right) \\
&= \int_V \partial_1 v(x)\,\tfrac{1}{t}\big(\psi(x-te) - \psi(x)\big)\,\mathrm{d}^n x \\
&\xrightarrow[t\to 0]{} -\int_V \partial_1 v(x)\partial_1\psi(x)\,\mathrm{d}^n x. \qquad (1.9)
\end{aligned}
$$

This limit is estimated using the coordinate expression of the functional A,

$$
\begin{aligned}
\left|\int_V \partial_1 v\,\partial_1\psi\right| &= \left|\int_V \tfrac{1}{g^{11}}\Big(\sum_{i,j} g^{ij}\partial_i v\,\partial_j\psi - \sum_{i\cdot j>1} g^{ij}\partial_i v\,\partial_j\psi\Big)\right| \\
&\leq \big\|\tfrac{1}{g^{11}}\big\|_\infty \big|A(u,\psi) - \langle\, u\,,\,\psi\,\rangle\big| + \sum_{i\cdot j>1}\left|\int_V \tfrac{g^{ij}}{g^{11}}\partial_i v\,\partial_j\psi\right| \\
&= \big\|\tfrac{1}{g^{11}}\big\|_\infty \big|\langle\, f-u\,,\,\psi\,\rangle\big| + \sum_{i\cdot j>1}\left|\int_V \partial_j\left(\tfrac{g^{ij}}{g^{11}}\partial_i v\right)\psi\right| \\
&\leq C'\left(\|f\|_2 + \|u\|_2 + \sum_{i\cdot j>1}\left\|\partial_j\left(\tfrac{g^{ij}}{g^{11}}\partial_i v\right)\right\|_2\right)\|\psi\|_2 \\
&\leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big)\|\psi\|_2. \qquad (1.10)
\end{aligned}
$$

Here the partial integration works just as in (1.9), and we have used the results on $\partial_j\partial_i v$ when either i or j differs from 1. The constants C', C are finite since the metric and its inverse are smooth and positive definite on the closure of V.

Now (1.10) holds for all $\psi \in \mathcal{C}^\infty(V)$ that are compactly supported in the interior of V, but these form a dense subset of $L^2(V)$. So the limit in (1.9) can be extended to a continuous linear functional on $L^2(V)$, i.e. there exists $F \in L^2(V)$

with $\|F\|_2 \leq C(\|f\|_2 + \|u\|_{W^{1,2}})$ such that for all $\psi \in L^2(V)$

$$\int_V \tfrac{1}{t}\big(\partial_1 v(x+te) - \partial_1 v(x)\big)\psi(x)\,\mathrm{d}^n x \;\underset{t\to 0}{\longrightarrow}\; \int_V F\psi.$$

Thus $\partial_1\partial_1 v = F$ weakly exists, lies in $L^2(V)$, and satisfies the estimate

$$\|\partial_1\partial_1 v\|_2 \leq C\big(\|f\|_2 + \|u\|_{W^{1,2}}\big).$$

This finishes the proof in the case $k = 0$.

Now we assume the theorem to hold for some $k \geq 0$, then the induction step is to first establish the claim for $k+1$ and then deduce the theorem for $k+1$.

Induction step for the claim ($k+1 \geq 1$):
Let $f \in W^{k+1,2}(M)$ and $u \in W^{k+2,2}(M)$ satisfy (1.2) and let X be as assumed in the claim. Then from the calculation (1.6) one obtains for all $\psi \in W^{1,2}(M)$

$$A(\mathcal{L}_X u\,,\,\psi) = \big\langle\, \mathcal{L}_X f + (f-u)\cdot \mathrm{div}X + {*\mathrm{d}}\big(\mathrm{HS}_0(\mathrm{d}u)\big)\,,\,\psi \,\big\rangle\,. \tag{1.11}$$

Indeed, in the first and second term of (1.6)

$$\begin{aligned}
-\langle\, f\,,\, D_{-t}\psi\,\rangle &= \tfrac{1}{t}\left[\int_M \phi_{-t}^*\big(\phi_t^* f\cdot\psi\cdot\phi_t^*\,\mathrm{dvol}_M\big) - \int_M f\cdot\psi\cdot\,\mathrm{dvol}_M\right]\\
&= \int_M D_t f\cdot\psi\cdot\,\mathrm{dvol}_M + \int_M \phi_t^* f\cdot\psi\cdot D_t\,\mathrm{dvol}_M\\
&\underset{t\to 0}{\longrightarrow} \int_M \mathcal{L}_X f\cdot\psi\cdot\,\mathrm{dvol}_M + \int_M f\cdot\psi\cdot\mathrm{div}X\cdot\,\mathrm{dvol}_M,
\end{aligned}$$

$$\begin{aligned}
&-\int_M \mathrm{d}\psi\wedge \mathrm{HS}_t(\mathrm{d}\phi_t^* u)\\
&= \int_M \psi\cdot\mathrm{d}\big(\mathrm{HS}_t(\mathrm{d}\phi_t^* u)\big)\\
&\quad - \tfrac{1}{t}\left[\int_{\partial M}\psi\cdot(\phi_t^*\mathrm{d}u)(\nu_{\phi_t^* g})\cdot\phi_t^*\,\mathrm{dvol}_{\partial M} - \int_{\partial M}\psi\cdot(\phi_t^*\mathrm{d}u)(\nu_g)\cdot\,\mathrm{dvol}_{\partial M}\right]\\
&\underset{t\to 0}{\longrightarrow} \int_M \psi\cdot\mathrm{d}\big(\mathrm{HS}_0(\mathrm{d}u)\big).
\end{aligned}$$

Here we used the fact from lemma 5.6 (i) that $*_g A|_{\partial M} = A(\nu_g)\,\mathrm{dvol}_{g,\partial M}$ for $A \in \Omega^1(M)$, where the dependence on the metric is indicated by subscripts. Both boundary terms vanish since ϕ_t preserves the boundary and unit normal $\nu = \nu_g$, so $(\phi_t^*\mathrm{d}u)(\nu_{\phi_t^* g}) = \mathrm{d}_{\phi_t} u(\nu_g) = 0$ and $(\phi_t^*\mathrm{d}u)(\nu_g) = \mathrm{d}_{\phi_t} u(\nu_g) = 0$.

The convergence as $t \to 0$ of the further terms in (1.6) to the terms claimed in (1.11) is due to $D_t u \to \mathcal{L}_X u$ in $W^{k+1,2}(M)$, $\phi_t^* u \to u$ in $W^{k+2,2}(M)$, and $D_t\,\mathrm{dvol}_M \to \mathrm{div}X\cdot\,\mathrm{dvol}_M$ uniformly.

This proves (1.11). Now the induction hypothesis (the theorem for k) implies that in fact $\mathcal{L}_X u \in W^{k+2,2}(M)$ and for some constants C and C'

$$\begin{aligned}\|\mathcal{L}_X u\|_{W^{k+2,2}} &\le C\big\|\mathcal{L}_X f + (f-u)\cdot \mathrm{div}X + *\mathrm{d}\big(\mathrm{HS}_0(\mathrm{d}u)\big)\big\|_{W^{k,2}} \\ &\le C'\big(\|f\|_{W^{k+1,2}} + \|u\|_{W^{k+2,2}}\big).\end{aligned}$$

In case $k+1 \ge 2$ this induction step can be proven by an even easier argument (which however requires $u \in W^{3,2}(M)$):

From remark 1.6 one sees that u is in fact a strong solution of (1.3). Now $\mathcal{L}_X u$ also meets the Neumann boundary condition:

$$\tfrac{\partial \mathcal{L}_X u}{\partial \nu}\big|_{\partial M} \;=\; \big(\mathcal{L}_X \tfrac{\partial u}{\partial \nu}\big)\big|_{\partial M} - \mathcal{L}_{[X,\nu]} u\big|_{\partial M} \;=\; 0.$$

This is due to the fact that $\frac{\partial u}{\partial \nu}$ vanishes on ∂M, X is tangential to ∂M, and its flow preserves ν, so the Lie bracket $[X,\nu] = \mathcal{L}_X \nu$ vanishes on ∂M.

Also observe that $[\Delta, \mathcal{L}_X]$ is a second order operator (this is clear from the local formula), and hence (1.3) provides

$$\Delta \mathcal{L}_X u + \mathcal{L}_X u \;=\; \mathcal{L}_X f + [\Delta, \mathcal{L}_X] u \;\in W^{k,2}(M).$$

So $\mathcal{L}_X u$ solves (1.3) with f replaced by $\mathcal{L}_X f + [\Delta, \mathcal{L}_X] u \in W^{k,2}(M)$. Then the induction hypothesis (the theorem for k) implies that $\mathcal{L}_X u \in W^{k+2,2}(M)$ and for some constants C, C'

$$\begin{aligned}\|\mathcal{L}_X u\|_{W^{k+2,2}} &\le C\big(\|\mathcal{L}_X f\|_{W^{k,2}} + \|[\Delta, \mathcal{L}_X] u\|_{W^{k,2}}\big) \\ &\le C'\big(\|f\|_{W^{k+1,2}} + \|u\|_{W^{k+2,2}}\big).\end{aligned}$$

Induction step for the theorem ($k+1 \ge 1$):

Assume the theorem to hold for $k \ge 0$ and let $f \in W^{k+1,2}(M)$. Then it is already known that $u \in W^{k+2,2}(M)$ and thus is a strong solution of (1.3). Now choose an atlas as in the case $k = 0$, then the task is to prove that $\partial_j \partial_\ell (u \circ \Phi_i^{-1})$ lies in $W^{k+1,2}(V_i)$ for all $i = 1, \ldots N$ and $j, \ell = 1, \ldots n$ and that for some finite constant C

$$\|\partial_j \partial_\ell (u \circ \Phi_i^{-1})\|_{W^{k+1,2}} \le C\big(\|f\|_{W^{k+1,2}} + \|u\|_{W^{k+2,2}}\big). \tag{1.12}$$

The estimate (1.4) then follows as for $k = 0$ from lemma E.3, the uniqueness of u, and the compactness of the embedding $W^{k+3,2}(M) \hookrightarrow W^{k+2,2}(M)$.

Also as for $k = 0$ the established claim (for $k+1$) fulfills this task in all cases except for when U_i intersects the boundary ∂M and $j = \ell = 1$. In that case denote $v := u \circ \Phi_i^{-1}$ and $V := \Phi_i(U_i) \subset \mathbb{R}^n$ and simply use the local formula for the Laplace operator,

$$\partial_1 \partial_1 v = \tfrac{1}{g^{11}}\Big(\Delta v - \sum_{i \cdot j > 1} g^{ij} \partial_i \partial_j v + \sum_{i,j,\ell} g^{ij} \Gamma_{ij}^{\ell} \partial_\ell v\Big).$$

Every term on the right hand side lies in $W^{k+1,2}(V)$: For the first term this is due to the assumption $\Delta v = f\circ\Phi_i^{-1} - v$. In the second term there are only second derivatives of v that were already covered by the claim and the first derivatives of v in the third term lie in $W^{k+1,2}(V)$ since $u \in W^{k+2,2}(M)$. Moreover, the metric, its inverse, and the Christoffel symbols are smooth and g^{11} is bounded away from zero since the inverse metric is positive definite. This proves that $\partial_1\partial_1 v \in W^{k+1,2}(V)$ and the estimate (1.12) also follows from the induction hypothesis and the claim:

$$\begin{aligned}
&\|\partial_1\partial_1 v\|_{W^{k+1,2}} \\
&\leq \left\|\tfrac{1}{g^{11}}\right\|_\infty \Big(\|f\circ\Phi_i^{-1} - v\|_{W^{k+1,2}} + \sum_{i\cdot j>1} \|g^{ij}\|_{W^{k+1,\infty}} \|\partial_i\partial_j v\|_{W^{k+1,2}} \\
&\qquad\qquad + \sum_{\ell} \Big\|\sum_{i,j} g^{ij}\Gamma_{ij}^{\ell}\Big\|_{W^{k+1,\infty}} \|\partial_\ell v\|_{W^{k+1,2}}\Big) \\
&\leq C\big(\|f\|_{W^{k+1,2}} + \|u\|_{W^{k+2,2}}\big)
\end{aligned}$$

for some finite constant C. This finishes the proof. □

Now all the hard work is done and we can prove the regularity theorem.

Proof of theorem 1.3 :
Let $u \in \mathcal{D}(M)$ and $f \in W^{k,2}(M)$ solve (wNP). By remark 1.2 (iii) we can assume $u \in W^{\ell,2}(M)$ for some $\ell \in \mathbb{Z}$ and then prove $u \in W^{k+2,2}(M)$ by bootstrapping ℓ. The equation for this procedure is provided by the assumption:

$$\langle\, u \,,\, \Delta\psi + \psi \,\rangle = \langle\, f + u \,,\, \psi \,\rangle \qquad \forall \psi \in \mathcal{C}_\nu^\infty(M). \tag{1.13}$$

Now depending on the regularity $u \in W^{\ell,2}(M)$ that is already established different arguments apply to prove that u is of even higher regularity, up to $W^{k+2,2}(M)$:

If $\ell \leq -2$ and $\ell \leq k$ then $f + u \in W^{\ell,2}(M)$ and thus

$$|\langle\, u \,,\, \Delta\psi + \psi \,\rangle| \leq \|f + u\|_{W^{\ell,2}} \|\psi\|_{W^{-\ell,2}} \qquad \forall \psi \in \mathcal{C}_\nu^\infty(M).$$

Now given any $\phi \in \mathcal{C}^\infty(M)$ we use theorem 1.8 to find $\psi \in \mathcal{C}_\nu^\infty(M)$ that solves $\Delta\psi + \psi = \phi$ and also meets $\|\psi\|_{W^{-\ell,2}} \leq C\|\phi\|_{W^{-\ell-2,2}}$ for some constant C. Hence $u \in W^{\ell+2,2}(M)$ since for some other constant C'

$$|\langle\, u \,,\, \phi \,\rangle| \leq C'\|\phi\|_{W^{-\ell-2,2}} \qquad \forall \phi \in \mathcal{C}^\infty(M).$$

In case $k \leq -2$ one can repeat this argument to obtain $u \in W^{k+2,2}(M)$, which finishes the proof. For $k \geq -1$ the iteration of the argument stops at $u \in L^2(M)$. Then the bootstrapping proceeds with the following argument starting at $\ell = -1$.

In case $-1 \leq \ell \leq k$ theorem 1.8 (with f replaced by $f + u \in W^{\ell,2}(M)$) provides $\tilde{u} \in W^{\ell+2,2}(M)$ such that

$$\langle\, \tilde{u} \,,\, \Delta\psi + \psi \,\rangle = \langle\, f + u \,,\, \psi \,\rangle \qquad \forall \psi \in \mathcal{C}_\nu^\infty(M).$$

But now as before $\Delta\psi + \psi$ runs through all of $\mathcal{C}^\infty(M)$ and by assumption u solves the same equation as $\tilde{u}$, hence

$$\langle\, \tilde{u} - u\,,\, \phi\,\rangle = 0 \qquad \forall \phi \in \mathcal{C}^\infty(M).$$

This implies $u = \tilde{u} \in W^{\ell+2,2}(M)$ and this argument can be repeated up to $\ell = k$ to prove $u \in W^{k+2,2}(M)$. □

As a first consequence of the regularity theorem we can determine the solutions of the homogeneous Neumann problem:

Corollary 1.9 *If $u \in \mathcal{D}(M)$ solves the homogeneous Neumann problem,* (wNP) *with $f = 0$, then u is a constant function on M. Conversely, every constant function solves the homogeneous Neumann problem.*

Proof: Assume $u \in \mathcal{D}(M)$ is a weak solution of the homogeneous Neumann problem, then theorem 1.3 implies that $u \in \bigcap_{k\in\mathbb{N}} W^{k,2}(M) = \mathcal{C}^\infty(M)$, and so by lemma N it actually is a strong solution. Thus Stokes' theorem can be applied to deduce that $\mathrm{d}u = 0$:

$$\|\mathrm{d}u\|_2^2 \;=\; \int_M \mathrm{d}u \wedge *\mathrm{d}u \;=\; \int_M u\,\Delta u\,\mathrm{dvol}_M + \int_{\partial M} u\,\tfrac{\partial u}{\partial \nu}\,\mathrm{dvol}_{\partial M} \;=\; 0.$$

Hence u must be constant. On the other hand, every constant function obviously solves the homogeneous Neumann problem in the strong sense. □

This corollary answers the question of uniqueness: every solution of the Neumann problem is unique up to adding a constant. Moreover, this uniqueness result is essential for the proof of the second estimate in theorem 1.4.

Proof of theorem 1.4 :
The first estimate follows directly from the estimate for the operator $\Delta + 1$: Every $u \in W^{k+2,2}(M)$ with $\frac{\partial u}{\partial \nu} = 0$ obviously is a solution of (1.3) with $f = \Delta u + u$. By remark 1.6 it also solves (1.2) and hence theorem 1.8 provides a finite constant C such that

$$\|u\|_{W^{k+2,2}} \;\leq\; C\|f\|_{W^{k,2}} \;\leq\; C\big(\|\Delta u\|_{W^{k,2}} + \|u\|_{W^{k,2}}\big).$$

This estimate can be improved by using the corresponding estimate for the term $\|u\|_{W^{k,2}}$ on the right hand side. But there will always remain some term in addition to $\|\Delta u\|_{W^{k,2}}$ due to the fact that the Laplace operator is not injective on the considered space of functions. This is the reason for the restriction to functions with zero mean value for the second estimate. For these corollary 1.9 provides the injectivity.

So let $X \subset W^{k+2,2}(M)$ be the subspace of functions with Neumann boundary condition and zero mean value. Then $\Delta : X \to W^{k,2}(M)$ is a bounded injective linear operator and the inclusion $X \hookrightarrow W^{k,2}(M)$ is compact by the Sobolev embedding theorem B.2. Now a consequence of the closed graph theorem, lemma E.3, implies the second estimate. $\square$

For the existence consider the Laplace operator as unbounded operator on $L^2(M)$ with domain

$$W^{2,2}_\nu(M) := \left\{ u \in W^{2,2}(M) \;\middle|\; \tfrac{\partial u}{\partial \nu} = 0 \right\}.$$

Proof of theorem 1.5 :
Firstly, if $u \in W^{k+2,2}(M)$ solves (NP) then by lemma N it also solves (wNP). Testing (wNP) with $\psi \equiv 1$ then yields $\int_M f = 0$.

In order to prove the sufficiency of this condition for the existence of a solution it suffices to consider the case $k = 0$. In case $k \geq 1$ the higher regularity of the unique solution then follows from theorem 1.3.

Firstly, the Laplace operator Δ with domain $W^{2,2}_\nu(M)$ is densely defined on $L^2(M)$. Indeed, the set of smooth functions with compact support in the interior of M is a subset of $W^{2,2}_\nu(M)$ and a dense subset of $L^2(M)$.

Secondly, our operator is symmetric: For every $u, v \in W^{2,2}_\nu(M)$ Stokes' theorem yields

$$\langle \Delta u \,,\, v \rangle \;=\; \int_M \mathrm{d}v \wedge *\mathrm{d}u - \int_{\partial M} \left(v \tfrac{\partial u}{\partial \nu} - u \tfrac{\partial v}{\partial \nu} \right) \mathrm{dvol}_{\partial M} \;=\; \langle \, u \,,\, \Delta v \, \rangle .$$

Next, the image $\mathrm{im}(\Delta)$ is closed. To see this consider a sequence $u_n \in W^{2,2}_\nu(M)$ such that Δu_n converges to some $f \in L^2(M)$. One can add a constant to every u_n to achieve $\int_M u_n = 0$ but not change the sequence Δu_n. Then the second estimate from theorem 1.4 implies that u_n is a Cauchy sequence in $W^{2,2}(M)$:

$$\|u_n - u_m\|_{W^{2,2}} \;\leq\; C \|\Delta u_n - \Delta u_m\|_2 \underset{m,n\to\infty}{\longrightarrow} 0.$$

Thus the u_n converge to some $u \in W^{2,2}(M)$. Due to the continuity of the embedding $W^{1,2}(M) \hookrightarrow L^1(\partial M)$ (theorem B.10) and by the continuity of the operator $\Delta : W^{2,2}(M) \to L^2(M)$, this limit solves $\frac{\partial u}{\partial \nu} = 0$ and $f = \Delta u$. Hence f also lies in $\mathrm{im}(\Delta) = \Delta(W^{2,2}_\nu(M))$.

The first two facts already imply $\ker(\Delta) = \mathrm{im}(\Delta)^\perp$: For $u \in W^{2,2}_\nu(M)$

$$\begin{aligned}
\Delta u = 0 \iff & \langle \Delta u \,,\, \psi \rangle = 0 \quad \forall \psi \in L^2(M) \\
\iff & \langle \Delta u \,,\, \psi \rangle = 0 \quad \forall \psi \in W^{2,2}_\nu(M) \\
\iff & \langle u \,,\, \Delta \psi \rangle = 0 \quad \forall \psi \in W^{2,2}_\nu(M) \\
\iff & u \in \mathrm{im}(\Delta)^\perp,
\end{aligned}$$

hence $\ker(\Delta) = \mathrm{im}(\Delta)^\perp \cap W_\nu^{2,2}(M)$. But $\mathrm{im}(\Delta)^\perp \cap W_\nu^{2,2}(M) = \mathrm{im}(\Delta)^\perp$ since every function orthogonal to $\mathrm{im}(\Delta)$ is a weak solution of the homogeneous Neumann problem and thus by corollary 1.9 automatically lies in $W_\nu^{2,2}(M)$.

Moreover, $\mathrm{im}(\Delta)$ is closed and $\mathrm{im}(\Delta)^\perp = \ker(\Delta)$, hence taking the orthogonal complement implies $\mathrm{im}(\Delta) = \ker(\Delta)^\perp$ and this proves the theorem: There exists a solution $u \in W_\nu^{2,2}(M)$ of $\Delta u = f$ if and only if the function f lies in $\mathrm{im}(\Delta) = \ker(\Delta)^\perp$, that is $\langle f , c \rangle = 0$ for all constants c, and this is just the condition that f has vanishing mean value, $\int_M f = 0$. □

We also note the following consequence for the Laplace operator.

Corollary 1.10 *The Hodge Laplace operator Δ with domain $W_\nu^{2,2}(M)$ is a self-adjoint operator on $L^2(M)$.*

Proof: In the preceding proof we have already established that this operator is densely defined and symmetric. This also implies that the domain of the dual operator

$$D^* = \left\{ u \in L^2(M) \,\middle|\, |\langle u , \Delta\psi \rangle| \le C\|\psi\|_2 \quad \forall \psi \in W_\nu^{2,2}(M) \right\}$$

contains the domain $D = W_\nu^{2,2}(M)$. But actually $D = D^*$ and thus the operator is selfadjoint. To see this note that for every $u \in D^*$ the density of $W_\nu^{2,2}(M) \subset L^2(M)$ implies that $\psi \mapsto \langle u , \Delta\psi \rangle$ extends to a continuous linear functional on $L^2(M)$. Hence there exists an $f \in L^2(M)$ such that $\langle u , \Delta\psi \rangle = \langle f , \psi \rangle$ for all $\psi \in \mathcal{C}_\nu^\infty(M)$. Then the regularity theorem 1.3 and lemma N imply that $u \in W_\nu^{2,2}(M)$. □

Chapter 2

$\mathbf{L^p}$-Theory

In this chapter we extend the existence and regularity results to L^p-spaces for $1 < p < \infty$.

Theorem 2.1 *Let $k \in \mathbb{N}_0$ and suppose that $u \in \mathcal{D}(M)$ is a weak solution of the Neumann problem* (wNP) *with $f \in W^{k,p}(M)$. Then $u \in W^{k+2,p}(M)$.*

Theorem 2.2 *Let $f \in W^{k,p}(M)$ for some $k \in \mathbb{N}_0$. Then there exists a solution $u \in W^{k+2,p}(M)$ of* (NP) *if and only if $\int_M f = 0$. This solution is unique up to an additive constant.*

These two theorems are corollaries of the L^2-results once the following elliptic estimate is established.

Theorem 2.3 *For every $k \in \mathbb{N}_0$ and $1 < p < \infty$ there exist constants C, C' such that for every $u \in W^{k+2,p}(M)$*

$$\begin{aligned} \|u\|_{W^{k+2,p}} &\leq C\big(\|\Delta u\|_{W^{k,p}} + \|u\|_{W^{k+1,p}}\big) && \text{if} \quad \tfrac{\partial u}{\partial \nu} = 0, \\ \|u\|_{W^{k+2,p}} &\leq C'\|\Delta u\|_{W^{k,p}} && \text{if} \quad \tfrac{\partial u}{\partial \nu} = 0 \quad \text{and} \quad \textstyle\int_M u = 0. \end{aligned}$$

Theorem 2.1 generalizes the L^2-regularity theorem 1.3 for $f \in W^{k,2}(M)$ with $k \in \mathbb{N}_0$. The generalization of the assumption $f \in W^{-k,2}(M)$ with $k \in \mathbb{N}$ to the case $p \neq 2$ is $f \in W^{-k,p}(M) := (W^{k,p^*}(M))^* \subset \mathcal{D}(M)$, where $p^* = \frac{p}{p-1}$ is the dual Sobolev exponent. This leads to the following extension of the theorems 2.1 and 2.3.

Theorem 2.3′ *For every $k \in \mathbb{N}$ and $1 < p < \infty$ there exists a constants C such that the following holds: Suppose that $u \in \mathcal{D}(M)$ is a weak solution of the Neumann problem (*wNP*) for $f \in W^{-k,p}(M)$. Then $u \in W^{-k+2,p}(M)$ and*

$$\|u\|_{W^{-k+2,p}} \leq C\big(\|f\|_{W^{-k,p}} + |\langle u\,,\,1\rangle|\big).$$

The proof of theorem 2.3' in case $k = 1$ will rely on theorem 4.3, which is proven independently of theorem 2.3'. In fact, there is no application of theorem 2.3' in this book – we only included it for the sake of completeness.

Remark 2.4

(i) In theorem $2.3'$ we denote by 1 the constant function $\psi \equiv 1$ on M. So if $u \in L^p(M)$, then $|\langle u, 1 \rangle| \leq (\mathrm{Vol}\, M)^{\frac{1}{p}^*} \|u\|_p$.

(ii) Let $u \in L^p(M)$ and suppose that for some constant c_u

$$\left| \int_M u \cdot \Delta\psi \right| \leq c_u \|\psi\|_{W^{1,p^*}} \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M).$$

Then u solves (wNP) for some $f \in W^{-1,p}(M)$ with $\|f\|_{W^{-1,p}} \leq c_u$ (to see this use the Hahn-Banach theorem, e.g. [R, Thm.5.16]). So theorem $2.3'$ asserts that $u \in W^{1,p}(M)$ and for some constant C

$$\|u\|_{W^{1,p}} \leq C\big(c_u + \|u\|_p\big).$$

Proof of theorem 2.3 :
By corollary 1.9 the Hodge Laplace operator is injective when restricted to the domain of functions $u \in W^{k+2,p}(M)$ that satisfy $\frac{\partial u}{\partial \nu} = 0$ and $\int_M u = 0$. Moreover, the embedding $W^{k+2,p}(M) \hookrightarrow W^{k+1,p}(M)$ is compact. Hence by lemma E.3 the second estimate follows from the first one. So it remains to prove the first estimate. We will first consider functions $u \in W^{2,p}(\mathbb{R}^n)$ with compact support and establish the Calderon-Zygmund inequality for general $W^{1,\infty}$-metrics on $\mathbb{R}^n$,

$$\|u\|_{W^{2,p}} \leq C\big(\|\Delta u\|_p + \|u\|_{W^{1,p}}\big). \tag{2.1}$$

We then use a patching argument to prove the theorem in the case $k = 0$. Finally, we establish the estimate for $k \geq 1$ by induction.

Step 1 (Euclidean metric on $\mathbb{R}^n$) : The Calderon-Zygmund inequality implies the estimate (2.1) with the Euclidean metric on $\mathbb{R}^n$: For $u \in W^{2,p}(\mathbb{R}^n)$ with compact support we obtain from theorem C.3

$$\|u\|_{W^{2,p}} \;\leq\; \|u\|_{W^{1,p}} + \|\nabla^2 u\|_p \;\leq\; \|u\|_{W^{1,p}} + C_0\|\Delta u\|_p.$$

The constant C_0 in the Calderon-Zygmund inequality only depends on $1 < p < \infty$.

Step 2 (Constant metric on $\mathbb{R}^n$) : We extend the estimate (2.1) for functions with compact support to all constant metrics $g \equiv (G_{ij})_{i,j=1,\dots,n}$:

Let ∇_g, Δ_g, and $\|\cdot\|_{g,W^{k,p}}$ denote the Levi-Civita connection, the Laplace operator, and the $W^{k,p}$-metric with respect to a metric g. The estimate in step 1 then holds for $\nabla_{\mathbb{1}}$, $\Delta_{\mathbb{1}}$ and $\|\cdot\|_{\mathbb{1},W^{k,p}}$. Since G is a symmetric, positive definite, real matrix, there is a linear diffeomorphism $\Psi:\mathbb{R}^n\to\mathbb{R}^n$ such that $\Psi^*g=\mathbb{1}$ (i.e. $\Psi^TG\Psi=\mathbb{1}$). Now consider $u\in W^{2,p}(\mathbb{R}^n)$ with compact support. Then step 1 applies to $u\circ\Psi$ (which obviously also is $W^{2,p}$-regular with compact support),

$$\|\nabla_g^2u\|_{g,L^p} \;=\; \|\nabla_{\Psi^*g}^2(u\circ\Psi)\|_{\Psi^*g,L^p} \;\le\; C_0\|\Delta_{\Psi^*g}(u\circ\Psi)\|_{\Psi^*g,L^p} \;=\; C_0\|\Delta_g u\|_{g,L^p}.$$

Here we have used the naturality of ∇ and Δ, that is $\nabla^2_{\Psi^*g}(u\circ\Psi)=\Psi^*(\nabla_g^2u)$ and $\Delta_{\Psi^*g}(u\circ\Psi)=(\Delta_g u)\circ\Psi$. Now the claimed estimate follows,

$$\|u\|_{g,W^{2,p}} \;\le\; \|u\|_{g,W^{1,p}}+\|\nabla_g^2u\|_{g,L^p} \;\le\; \|u\|_{g,W^{1,p}}+C_0\|\Delta_g u\|_{g,L^p}.$$

Step 3 (Metric L^∞-close to a constant metric on $\mathbb{R}^n$) :

Let $g_0\in\mathbb{R}^{n\times n}$ be some constant metric on $\mathbb{R}^n$. Then we find $\varepsilon>0$ such that for all compact subsets $B\subset\mathbb{R}^n$ and for all metrics $g\in W^{1,\infty}(B,\mathbb{R}^{n\times n})$ with $\|g-g_0\|_{g_0,L^\infty(B)}\le\varepsilon$ there exists a constant C such that the (2.1) holds for all $u\in W^{2,p}(\mathbb{R}^n)$ with $\operatorname{supp}u\subset B$.

Firstly, we choose $\varepsilon>0$ such that $\|\sqrt{\det g}-\sqrt{\det g_0}\|_{L^\infty(B)}\le\frac12\sqrt{\det g_0}$ and thus

$$\tfrac12\sqrt{\det g}\;\le\;\sqrt{\det g_0}\;\le\;2\sqrt{\det g}.$$

This ensures that the L^p-norms on B with respect to g and g_0 are equivalent: For every $f\in L^p(\mathbb{R}^n)$

$$\tfrac12\|f\|_{g,L^p(B)}\;\le\;\|f\|_{g_0,L^p(B)}\;\le\;2\|f\|_{g,L^p(B)}. \tag{2.2}$$

In the following, all Sobolev norms are to be understood as taken over B. The above equivalence actually does not require g to be close to g_0 if we fix B and allow for any constant instead of 2. However, the control of $|g-g_0|$ on B is crucial for comparing the Laplace operators for the two metrics:

$$\begin{aligned}\big|\Delta_g u-\Delta_{g_0}u\big| &= \big|\big(g^{ij}-g_0^{ij}\big)\partial_i\partial_j u-g^{ij}\Gamma_{ij}^k\partial_k u\big|\\ &\le \big\|g^{-1}-g_0^{-1}\big\|_{g_0,L^\infty(B)}\big|\nabla_{g_0}^2u\big|_{g_0}+C\big|\nabla u\big|_{g_0}.\end{aligned}$$

Here ∇_g and $|\cdot|_g$ denote the Levi-Civita connection and the pointwise norm on tensors with respect to a metric g respectively. Note that for functions $\nabla_g u=\mathrm{d}u$ as well as $|u|_g$ is actually independent of the metric, so in that case we drop the subscript indicating the metric. Moreover, the Christoffel symbols Γ_{ij}^k for g are bounded on B, giving rise to the constant C, and the Christoffel symbols for g_0 vanish. We now insert $\Delta_g u$ into the estimate from step 2,

$$\begin{aligned}&\big\|\nabla_{g_0}^2u\big\|_{g_0,L^p}\\ &\le C_0\big\|\Delta_{g_0}u\big\|_{g_0,L^p}\\ &\le C_0\big\|\Delta_g u\big\|_{g_0,L^p}+C_0\big\|\Delta_g u-\Delta_{g_0}u\big\|_{g_0,L^p}\\ &\le C_0\big\|\Delta_g u\big\|_{g_0,L^p}+C_0\big(\big\|g^{-1}-g_0^{-1}\big\|_{g_0,L^\infty}\big\|\nabla_{g_0}^2u\big\|_{g_0,L^p}+C\big\|\nabla u\big\|_{g_0,L^p}\big).\end{aligned}$$

Now it is crucial to choose $\varepsilon > 0$ such that $C_0\big\|g^{-1}-g_0^{-1}\big\|_{g_0,L^\infty(B)} \leq \frac{1}{2}$, since then this inequality can be rearranged to

$$\big\|\nabla_{g_0}^2 u\big\|_{g_0,L^p} \leq 2C_0\big\|\Delta_g u\big\|_{g_0,L^p} + 2C_0 C\big\|\nabla u\big\|_{g_0,L^p}. \tag{2.3}$$

Next, it is again enough to have a $W^{1,\infty}$-bound on g in order to estimate the following norm of 2-tensors on B,

$$\big|\nabla_g^2 u - \nabla_{g_0}^2 u\big|_g \;=\; \big|\Gamma_{ij}^k \partial_k u\big|_g \;\leq\; C\big|\nabla u\big|_g.$$

As before, the constant C depends on g. We then obtain

$$\big\|\nabla_g^2 u\big\|_{g,L^p} \;\leq\; \big\|\nabla_{g_0}^2 u\big\|_{g,L^p} + C\big\|\nabla u\big\|_{g,L^p} \;\leq\; 4\big\|\nabla_{g_0}^2 u\big\|_{g_0,L^p} + C\big\|\nabla u\big\|_{g,L^p}. \tag{2.4}$$

Here the second step uses (2.2) and the equivalence of the pointwise norms on 2-tensors,

$$\big|\nabla_{g_0}^2 u\big|_g \leq 2\big|\nabla_{g_0}^2 u\big|_{g_0}.$$

This again requires a small choice of $\varepsilon > 0$, which however is independent of B. Similarly, we will have

$$|\nabla u|_{g_0} \leq 2|\nabla u|_g.$$

Finally, we put (2.3) and (2.4) together to obtain for all $u \in W^{2,p}(\mathbb{R}^n)$ with support $\operatorname{supp} u \subset B$

$$\begin{aligned}
\|u\|_{g,W^{2,p}} &\leq \big\|\nabla_g^2 u\big\|_{g,L^p} + \|u\|_{g,W^{1,p}} \\
&\leq 4\big\|\nabla_{g_0}^2 u\big\|_{g_0,L^p} + (C+1)\|u\|_{g,W^{1,p}} \\
&\leq 8C_0\big\|\Delta_g u\big\|_{g_0,L^p} + 8C_0 C\big\|\nabla u\big\|_{g_0,L^p} + (C+1)\|u\|_{g,W^{1,p}} \\
&\leq C'\big(\big\|\Delta_g u\big\|_{g,L^p} + \|u\|_{g,W^{1,p}}\big).
\end{aligned}$$

The last step again uses (2.2), and the finite constant C' will depend on g.

Step 4 (General metric on $\mathbb{R}^n$) :

For every compact subset $K \subset \mathbb{R}^n$ equipped with a metric $g \in W^{1,\infty}(K, \mathbb{R}^{n\times n})$ we find a constant C such that (2.1) holds for all $u \in W^{2,p}(\mathbb{R}^n)$ with support $\operatorname{supp} u \subset K$.

Fix any $x \in K$ and let $\varepsilon > 0$ be the constant from step 3 for the metric $g_0 = g(x)$. Since the metric g is continuous, we then have $\|g - g(x)\|_{g(x),L^\infty} \leq \varepsilon$ on some open Euclidean ball $B_{\delta(x)}(x) \subset \mathbb{R}^n$ with $\delta(x) > 0$. The compact set K is covered by finitely many of these balls, $K \subset B_1 \cup \ldots \cup B_N$. For each ball B_i, the estimate (2.1) holds by step 3 with some constant C_i for all $u \in W^{2,p}(\mathbb{R}^n)$ with $\operatorname{supp} u \subset B_i$.

Next, we choose a partition of unity $\phi_i \in \mathcal{C}^\infty(\mathbb{R}^n, [0,1]))$, $\operatorname{supp}\phi_i \subset B_i$ such that $\sum_{i=1}^N \phi_i\big|_K \equiv 1$. Then for any $u \in W^{2,p}(\mathbb{R}^n)$ and $i = 1, \ldots, N$ step 3 applies to $\phi_i u \in W^{2,p}(\mathbb{R}^n)$ and yields

$$\begin{aligned}\|\phi_i u\|_{W^{2,p}(B_i)} &\le C_i \left(\|\Delta(\phi_i u)\|_{L^p(B_i)} + \|\phi_i u\|_{W^{1,p}(B_i)}\right) \\ &\le C_i' \left(\|\Delta u\|_{L^p(K)} + \|u\|_{W^{1,p}(K)}\right).\end{aligned}$$

Here all norms and operators are defined with respect to the metric g, and the new constants C_i' depend on $\|\phi_i\|_{W^{2,\infty}}$. Summing this up yields the claim with $C = \sum_{i=1}^N C_i'$,

$$\|u\|_{W^{2,p}(K)} \le \sum_{i=1}^N \|\phi_i u\|_{W^{2,p}(B_i)} \le C \left(\|\Delta u\|_{L^p(K)} + \|u\|_{W^{1,p}(K)}\right).$$

Step 5 (Proof in case $k = 0$) :

Since M is compact it is covered by finitely many coordinate charts $\psi_i : U_i \to M$ for $i = 1, \dots, N$. In the interior, the domains U_i of the coordinate charts can be chosen as the open unit ball $U_i = D^n \subset \mathbb{R}^n$. At the boundary, the coordinate charts U_i can be chosen as $U_i = [0,1) \times D^{n-1} \subset \mathbb{R}^n$. This construction uses a coordinate chart of ∂M defined on $D^{n-1} \subset \mathbb{R}^{n-1}$ and combines it whith the flow of a vector field $-\tilde{\nu}$ that extends $-\nu$. If we denote the normal coordinate by $x^0 \in [0,1)$, then this construction maps $\{x_0 = 0\}$ to ∂M, and the pullback of the outer unit normal is $\psi^*\nu = -\frac{\partial}{\partial x^0}$. In both cases we equip U_i with the (smooth) pullback metric $\psi_i^* g$ induced by the metric on M.

Moreover, we choose a subordinate partition of unity $\phi_i \in \mathcal{C}^\infty(M, [0,1])$, $\operatorname{supp} \phi_i \subset \psi_i(U_i)$ such that $\sum_{i=1}^N \phi_i \equiv 1$ on M. This can be constructed such that $\frac{\partial}{\partial \nu}\phi_i = 0$: We start off with a partition of unity on ∂M subordinate to the cover by $\psi_i(U_i) \cap \partial M$. These functions can be extended constantly along the flowlines of $\tilde{\nu}$ to give a partition of unity in a tubular neighbourhood of ∂M, supported in the $\psi_i([0,1) \times D^{n-1}$. One can then interpolate via a cutoff function on M between this partition and any given partition of unity on the the cover of M by all $\psi_i(U_i)$.

Now let a function $u \in W^{2,p}(M)$ with $\frac{\partial}{\partial \nu} u = 0$ be given. We write it as $u = \sum_{i=1}^N \phi_i u$ and note that $\psi_i^*(\phi_i u) \in W^{2,p}(U_i)$. In the interior case $U_i = D_1$ this function is supported in the open ball D_1 and thus extends trivially to a $W^{2,p}$-function on $\mathbb{R}^n$. So we apply step 4 with the metric $\psi_i^* g$ to obtain

$$\begin{aligned}\|\phi_i u\|_{W^{2,p}(\psi_i(U_i))} &= \|\psi_i^*(\phi_i u)\|_{\psi_i^* g, W^{2,p}(U_i)} \\ &\le C_i \left(\|\Delta_{\psi_i^* g} \psi_i^*(\phi_i u)\|_{\psi_i^* g, L^p(U_i)} + \|\psi_i^*(\phi_i u)\|_{\psi_i^* g, W^{1,p}(U_i)}\right) \\ &\le C_i' \left(\|\Delta u\|_{L^p(\psi_i(U_i))} + \|u\|_{W^{1,p}(\psi_i(U_i))}\right). \qquad (2.5)\end{aligned}$$

Here the final constant C_i' includes a bound on the first and second derivatives of the cutoff function $\phi_i \in \mathcal{C}^\infty(M)$.

In the boundary case, when $U_i = [0,1) \times D^{n-1}$, we need to use the boundary condition on u, which becomes $\frac{\partial}{\partial x^0}(\psi_i^* u)\big|_{x_0=0} = 0$ in the coordinates. Due to the appropriate construction of the ϕ_i we also have

$$\frac{\partial}{\partial x^0}(\psi_i^*(\phi_i u))\Big|_{x_0=0} \;=\; -\Big(\frac{\partial \phi_i}{\partial \nu}u + \phi_i\frac{\partial u}{\partial \nu}\Big)\circ\psi_i\Big|_{x_0=0} \;=\; 0.$$

This allows us to extend $\psi_i^*(\phi_i u)$ across the boundary as follows: We denote the coordinates by $(x_0, x) \in (-1,1)\times D^{n-1}$ and introduce the reflection

$$\tau: \begin{array}{ccc} (-1,0]\times D^{n-1} & \longrightarrow & [0,1)\times D^{n-1} \\ (x_0,x) & \longmapsto & (-x_0,x) \end{array}$$

We then extend $\psi_i^*(\phi_i u) \in W^{2,p}([0,1)\times D^{n-1})$ to $\tilde{u}_i \in W^{2,p}((-1,1)\times D^{n-1})$ by

$$\tilde{u}_i(x_0,x) = \begin{cases} \psi_i^*(\phi_i u)\,(x_0,x), & x_0 \geq 0 \\ \tau^*\psi_i^*(\phi_i u)\,(x_0,x) = \psi_i^*(\phi_i u)\,(-x_0,x), & x_0 \leq 0. \end{cases}$$

This indeed defines a $W^{2,p}$-regular function since $\frac{\partial}{\partial x^0}\tilde{u}_i\big|_{x_0=0} = 0$ is welldefined. Now $\tilde{u}_i$ is supported in the open domain $(-1,1)\times D^{n-1} \subset \mathbb{R}^n$, so we could apply the estimate in step 4 to it. For that purpose we also extend the metric $\psi_i^* g$ to a metric $\tilde{g}_i \in W^{1,\infty}((-1,1)\times D^{n-1}, \mathbb{R}^{n\times n})$ by the same reflection,

$$\tilde{g}_i(x_0,x) = \begin{cases} \psi_i^* g\,(x_0,x), & x_0 \geq 0 \\ \tau^*\psi_i^* g\,(x_0,x), & x_0 \leq 0. \end{cases}$$

To see that this metric is $W^{1,\infty}$-regular we only need to check its continity at $x_0 = 0$. This is automatic from the construction except for the pairing of $\frac{\partial}{\partial x^0}$ with tangent vectors X to D^{n-1}. For that pairing recall that ψ_i maps $\{0\}\times D^{n-1}$ to ∂M, which is perpendicular to ν. So we indeed obtain continuity:

$$\psi_i^* g(\tfrac{\partial}{\partial x^0}, X) \;=\; g(-\nu, (\psi_i)_* X) \;=\; 0 \;=\; g(\nu, (\psi_i)_* X) \;=\; \tau^*\psi_i^* g(\tfrac{\partial}{\partial x^0}, X).$$

We can thus apply the estimate in step 4 on $K = (-1,1)\times D^{n-1}$ with the metric $\tilde{g}_i$ to the function $\tilde{u}_i$. Dropping the subscript i and denoting $D = D^{n-1}$ this yields

$$\begin{aligned}
&\|\tilde{u}\|_{\tilde{g},W^{2,p}(K)} \\
&\leq C\left(\|\Delta_{\tilde{g}}\tilde{u}\|_{\tilde{g},L^p(K)} + \|\tilde{u}\|_{\tilde{g},W^{1,p}(K)}\right) \\
&\leq C\Big(\|\Delta_{\psi^* g}\psi^*(\phi u)\|_{\psi^* g,L^p((0,1)\times D)} + \|\Delta_{\tau^*\psi^* g}\tau^*\psi^*(\phi u)\|_{\tau^*\psi^* g,L^p((-1,0)\times D)} \\
&\qquad + \|\psi^*(\phi u)\|_{\psi^* g,W^{1,p}((0,1)\times D)} + \|\tau^*\psi^*(\phi u)\|_{\tau^*\psi^* g,W^{1,p}((-1,0)\times D)}\Big) \\
&= 2C\left(\|\Delta(\phi u)\|_{L^p(\psi(U))} + \|\phi u\|_{W^{1,p}(\psi(U))}\right) \\
&\leq C'\left(\|\Delta u\|_{L^p(\psi(U))} + \|u\|_{W^{1,p}(\psi(U))}\right).
\end{aligned}$$

Here the constant C' depends on the choice of the cutoff function ϕ. Thus we obtain on all coordinate patches $U_i = [0,1) \times D$ at the boundary

$$\begin{aligned}
&\|\phi_i u\|_{W^{2,p}(\psi_i(U_i))} \\
&= \tfrac{1}{2}\left(\|\psi_i^*(\phi_i u)\|_{\psi_i^* g, W^{2,p}((0,1)\times D)} + \|\tau^*\psi_i^*(\phi_i u)\|_{\tau^*\psi_i^* g, W^{2,p}((-1,0)\times D)}\right) \\
&= \tfrac{1}{2}\|\tilde{u}\|_{\tilde{g}, W^{2,p}((-1,1)\times D)} \\
&\leq C_i'\left(\|\Delta u\|_{L^p(\psi_i(U_i))} + \|u\|_{W^{1,p}(\psi_i(U_i))}\right). \qquad (2.6)
\end{aligned}$$

Now we can sum up over all coordinate patches, using (2.5) and (2.6) respectively for the interior and boundary patches,

$$\|u\|_{W^{2,p}(M)} \;\leq\; \sum_{i=1}^{N} \|\phi_i u\|_{W^{2,p}(\psi_i(U_i))} \;\leq\; C\left(\|\Delta u\|_{L^p(M)} + \|u\|_{W^{1,p}(M)}\right).$$

This proves the first estimate in the theorem for $k = 0$ with the constant $C = \sum_{i=1}^N C_i'$. The second estimate follows from the injectivity of the operator as explained in the beginning.

Step 6 (Proof in case $k \geq 1$) :

For $k \geq 1$ the theorem now follows by induction. It remains to prove the first estimate and here the induction step works as in the proof of theorem 1.8 so we only give a sketch:

Assume the estimate to hold for some $\ell \geq 0$. First consider vector fields X on M that are tangential to ∂M and satisfy $[\mathcal{L}_X, \mathcal{L}_\nu] = 0$. (This holds when the flow of X preserves ν since $[\mathcal{L}_X, \mathcal{L}_\nu] = \mathcal{L}_{\mathcal{L}_X \nu}$.) Then for $u \in W^{\ell+3,p}(M)$ with $\frac{\partial u}{\partial \nu} = 0$ one also has $\frac{\partial}{\partial \nu}\mathcal{L}_X u = 0$, and since $[\Delta, \mathcal{L}_X]$ is a second order operator one obtains from the induction hypothesis

$$\begin{aligned}
\|\mathcal{L}_X u\|_{W^{\ell+2,p}} &\leq C\big(\|\Delta \mathcal{L}_X u\|_{W^{\ell,p}} + \|u\|_{W^{\ell+1,p}}\big) \\
&\leq C_X\big(\|\Delta u\|_{W^{\ell+1,p}} + \|u\|_{W^{\ell+2,p}}\big).
\end{aligned}$$

The constant C_X depends on X. This estimate can be applied to X being any cut off coordinate vector field except for the normal direction on charts near the boundary. On these charts it remains to consider the $\ell+3$-rd derivative of u in the normal direction. Here one can use the local formula of Δu to express the second normal derivative of u by Δu and other derivatives of u for which the estimate was already established. Thus one obtains the estimate for $k = \ell + 1$,

$$\|u\|_{W^{\ell+3,p}} \leq C\big(\|\Delta u\|_{W^{\ell+1,p}} + \|u\|_{W^{\ell+2,p}}\big).$$

□

Proof of theorem 2.2 :
The necessity of the condition $\int_M f = 0$ for the existence of a solution of the Neumann problem follows as in the case $p = 2$: If $u \in W^{k+2,p}(M)$ solves (NP) then by lemma N it also solves (wNP), which (tested with $\psi \equiv 1$) yields $\int_M f = 0$.

In order to prove the sufficiency of that condition let $f \in W^{k,p}(M)$ be given such that $\int_M f = 0$. Choose a sequence $\tilde{f}_i \in \mathcal{C}^\infty(M)$ that converges to f in the L^p-norm. Then also $\int_M \tilde{f}_i$ converges to $\int_M f = 0$ since M has finite volume. Thus

$$f_i := \tilde{f}_i - \tfrac{1}{\mathrm{Vol}(M)} \int_M \tilde{f}_i \in \mathcal{C}^\infty(M)$$

is a sequence of functions with vanishing mean value that still converges to f in the L^p-norm. Then the L^2-theorems 1.5 and 1.3 provide solutions $u_i \in \mathcal{C}^\infty(M)$ of the Neumann problem (NP) with f replaced by f_i. We can choose the u_i to have vanishing mean value such that theorem 2.3 provides

$$\|u_i - u_j\|_{W^{2,p}} \leq C\|\Delta u_i - \Delta u_j\|_p = C\|f_i - f_j\|_p \underset{i,j\to\infty}{\longrightarrow} 0.$$

Thus these u_i converge to some $u \in W^{2,p}(M)$. The limit solves $\Delta u = f$ due to the continuity of $\Delta : W^{2,p}(M) \to L^p(M)$ and theorem B.10 implies that u also meets the Neumann boundary condition. Uniqueness follows from corollary 1.9. □

Proof of theorem 2.1 :
From the already established theorem 2.2 we obtain a solution $\tilde{u} \in W^{k+2,p}(M)$ of the Neumann problem (NP) for any given $f \in W^{k,p}(M)$. By lemma N this also is a weak solution, hence for any other weak solution $u \in \mathcal{D}(M)$

$$\langle u - \tilde{u} \,,\, \Delta\psi \rangle = 0 \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M).$$

Now for every $\phi \in \mathcal{C}^\infty(M)$ with $\int_M \phi = 0$ there exists a $\psi \in \mathcal{C}^\infty_\nu(M)$ with $\Delta\psi = \phi$ and hence

$$\langle u - \tilde{u} - C \,,\, \phi \rangle = 0.$$

Here we can choose the finite constant $C := \frac{1}{\mathrm{Vol} M} \langle u - \tilde{u} \,,\, 1 \rangle$ such that the equality holds in fact for all $\phi \in \mathcal{C}^\infty(M)$. Indeed, every constant can be slotted in as long as $\int_M \phi = 0$. With the special constant the equality then also holds for ϕ equalling any constant. But every integrable function can be written as a sum of a constant and a function with vanishing mean value. Now this proves that $u = \tilde{u} + C \in W^{k+2,p}(M)$. □

Proof of theorem 2.3′ :
In case $k \geq 2$ this theorem is a corollary of the theorems 2.2 and 2.3 : Every $\phi \in \mathcal{C}^\infty(M)$ can be written as $\phi = \Delta\psi + c_\phi$, where $c_\phi = (\mathrm{Vol}\, M)^{-1}\int_M \phi$ and $\psi \in \mathcal{C}^\infty_\nu(M)$ such that $\|\psi\|_{W^{k,p^*}} \leq C'\|\phi - c_\phi\|_{W^{k-2,p^*}}$. Let $c := \langle\, u, 1\,\rangle$ then for some constant C

$$\begin{aligned} |\langle\, u-c\,,\,\phi\,\rangle| \;&=\; |\langle\, u-c\,,\,\phi - c_\phi\,\rangle| \;=\; \left|\langle\, u\,,\,\Delta\psi\,\rangle - c\textstyle\int_M \Delta\psi\right| \;=\; |\langle\, f\,,\,\psi\,\rangle| \\ &\leq\; C\|f\|_{(W^{k,p^*}(M))^*}\|\phi\|_{W^{k,p^*}}. \end{aligned}$$

This inequality extends to all $\phi \in W^{k,p^*}(M)$ and thus proves that $u - c$ and hence also u lie in $\left(W^{k,p^*}(M)\right)^*$ with

$$\begin{aligned} \|u\|_{(W^{k,p^*}(M))^*} &\leq \|u-c\|_{(W^{k,p^*}(M))^*} + \|c\|_{(W^{k,p^*}(M))^*} \\ &\leq C\|f\|_{(W^{k,p^*}(M))^*} + (\mathrm{Vol}\, M)^{\frac{1}{p}}|\,\langle\, u\,,\,1\,\rangle\,|. \end{aligned}$$

In case $k = 1$ the theorem can be proven in the same way using the fact that for $\phi \in \mathcal{C}^\infty(M)$ with $\int_M \phi = 0$ there exists a solution $\psi \in \mathcal{C}^\infty_\nu(M)$ of $\Delta\psi = \phi$ with $\|\psi\|_{W^{1,p^*}} \leq C\|\phi\|_{(W^{1,p^*}(M))^*}$. We do not prove that here, however, it follows from theorem 4.3 [1], which is proven independently of theorem 2.3’. □

Again, the existence result can be reformulated as a statement about the Laplace operator.

Corollary 2.5 *For all $k \in \mathbb{N}_0$ and $1 < p < \infty$ the Hodge Laplace operator Δ with domain*

$$W^{k+2,p}_\nu(M) := \left\{ u \in W^{k+2,p}(M) \;\middle|\; \tfrac{\partial u}{\partial \nu} = 0 \right\}.$$

and range $W^{k,p}(M)$ is a Fredholm operator of index 0.

Proof: By corollary 1.9 the kernel is finite dimensional and in fact just consists of the constants. Theorem 2.2 asserts that the image is closed and that the cokernel is isomorphic to the constants, too. Indeed, the vanishing of the mean value is a closed condition on $W^{k,p}(M)$, and given any function in $W^{k,p}$ one only has to add a constant to achieve mean value zero, i.e. a function in the image of Δ. Thus the operator is Fredholm with index 0. □

[1] Theorem 4.7 asserts that $\mathrm{im}\,\nabla' = \mathrm{im}\,\nabla'\nabla$ is closed. In this case $(\mathrm{im}\,\nabla')^\perp = H^0(M,\nabla)$ are the constants and thus $\phi \in \mathrm{im}\,\nabla'$.

Chapter 3

Inhomogeneous Boundary Conditions

In this chapter we consider the Neumann problem with inhomogeneous boundary conditions,

$$\begin{cases} \Delta u = f & \text{on } M, \\ \frac{\partial u}{\partial \nu} = g & \text{on } \partial M. \end{cases} \tag{3.1}$$

Throughout this chapter let $1 < p < \infty$ and $k \in \mathbb{N}_0$. If f lies in $W^{k,p}(M)$ then the natural space for the boundary values g is

$$W_\partial^{k+1,p}(M) := \big\{ G|_{\partial M} \;\big|\; G \in W^{k+1,p}(M) \big\} \cong W^{k+1,p}(M) \,/\, W_\delta^{k+1,p}(M).$$

Here $W_\delta^{k+1,p}(M) \subset W^{k+1,p}(M)$ is defined as the $W^{k+1,p}$-closure of the set of smooth functions on M that vanish on the boundary ∂M. We equip $W_\partial^{k+1,p}(M)$ with the quotient norm

$$\|g\|_{W_\partial^{k+1,p}} = \inf\big\{ \|G\|_{W^{k+1,p}} \;\big|\; G \in W^{k+1,p}(M), G|_{\partial M} = g \big\}.$$

This makes $W_\partial^{k+1,p}(M)$ a Banach space. (The quotient of a Banach space by a closed subspace is indeed complete with respect to the quotient norm.) A necessary condition for the existence of a solution $u \in W^{2,p}(M)$ of (3.1) is

$$\int_M f + \int_{\partial M} g = 0. \tag{3.2}$$

This is due to Stokes' theorem (of course, Stokes' theorem only applies to smooth functions, but this is overcome by smooth approximation of u as for lemma N):

$$\int_M \Delta u \,\mathrm{dvol}_M \;=\; -\int_M \mathrm{d} * \mathrm{d}u \;=\; -\int_{\partial M} *\mathrm{d}u \;=\; -\int_{\partial M} \tfrac{\partial u}{\partial \nu} \,\mathrm{dvol}_{\partial M}.$$

The main goal of this chapter is to prove that this condition is in fact sufficient.

Theorem 3.1 *Let $f \in L^p(M)$ and $g \in W^{1,p}_\partial(M)$. Then there exists a solution $u \in W^{2,p}(M)$ of (3.1) if and only if (3.2) holds. This solution is unique up to an additive constant.*

This essentially follows from the existence theorem for the Neumann problem with homogeneous boundary condition, however there is one crucial point:

One way of solving the problem is to first solve it for smooth f_i and g_i that meet (3.2) and converge to f and g. The solutions u_i exist since for $g_i \in \mathcal{C}^\infty(\partial M)$ one easily finds a $v_i \in \mathcal{C}^\infty(M)$ that satisfies the boundary condition. Then one solves the Neumann problem (NP) with f replaced by $f_i - \Delta v_i \in \mathcal{C}^\infty(M)$. It remains to prove that these solutions u_i (that can in addition be chosen to satisfy $\int_M u_i = 0$) converge to a solution of (3.1) in $W^{2,p}(M)$. For this purpose one needs the estimate

$$\|u\|_{W^{2,p}} \leq C\left(\|\Delta u\|_p + \left\|\tfrac{\partial u}{\partial \nu}\right\|_{W^{1,p}_\partial}\right) \tag{3.3}$$

for all $u \in W^{2,p}(M)$ with $\int_M u = 0$. This was proven by Agmon, Douglis, and Nirenberg in their general treatment of elliptic boundary value problems, see [ADN, Thm.15.2] .

The alternative approach is to first find a function $v \in W^{2,p}(M)$ that satisfies the boundary condition $\frac{\partial v}{\partial \nu} = g$ and then solve the Neumann problem (NP) with f replaced by $f - \Delta v \in L^2(M)$. But to establish the existence of v is no easier than proving (3.3). It seems that the easiest proof is to use the explicit solution of the Neumann problem with inhomogeneous boundary conditions on the half space, that is convolution with the Poisson kernel. We will give this proof, from this also deduce (3.3), and more generally obtain the following regularity result.

Theorem 3.2 (Agmon, Douglis, Nirenberg)
Let $f \in W^{k,p}(M)$ and $g \in W^{k+1,p}_\partial(M)$ for some $k \in \mathbb{N}_0$. Assume that $u \in \mathcal{D}(M)$ is a weak solution of (3.1), that is

$$\langle\, u\,,\, \Delta\psi\,\rangle = \int_M f\,\psi + \int_{\partial M} g\,\psi \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M). \tag{3.4}$$

Then $u \in W^{k+2,p}(M)$ and this actually solves (3.1) in the strong sense. Moreover, there exist constants C, C' such that for all $u \in W^{k+2,p}(M)$

$$\|u\|_{W^{k+2,p}} \leq C'\left(\|\Delta u\|_{W^{k,p}} + \left\|\tfrac{\partial u}{\partial \nu}\right\|_{W^{k+1,p}_\partial} + \|u\|_{W^{k+1,p}}\right),$$

$$\|u\|_{W^{k+2,p}} \leq C\left(\|\Delta u\|_{W^{k,p}} + \left\|\tfrac{\partial u}{\partial \nu}\right\|_{W^{k+1,p}_\partial}\right) \qquad \text{if } \int_M u = 0.$$

Remark 3.3

(i) In this theorem f and g automatically satisfy (3.2) – this follows from testing the weak equation (3.4) with $\psi \equiv 1$.

(ii) As in lemma N every strong solution $u \in W^{2,p}(M)$ of (3.1) also solves the weak equation (3.4).

(iii) We will see in the proof of theorem 3.2 that the optimal constants C and C' for $k \in \mathbb{N}_0$ depend continuously on the metric of M: For C' this dependence is with respect to the $W^{k+1,\infty}$-topology, for C one has to use the $W^{k+2,\infty}$-topology on the space of metrics.

The key to the previously described results is theorem C, which we now restate and prove.

Theorem 3.4 (Agmon, Douglis, Nirenberg)
Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Then there is a constant C such that for every $G \in W^{k+1,p}(M)$ there exists $u \in W^{k+2,p}(M)$ that satisfies

$$\tfrac{\partial u}{\partial \nu} = G\big|_{\partial M}, \qquad \|u\|_{W^{k+2,p}} \leq C\|G\|_{W^{k+1,p}}. \tag{3.5}$$

Proof: In a first step we will give an explicit construction on a bounded part of the Euclidean half space. This will then be used for a patching construction in the general case.

1.) Local construction:
Let $a > 0$, let $V \subset \mathbb{R}^n$ be a bounded domain, and equip the subset $[0,a) \times V$ of the half space $\mathbb{H}^{n+1} = [0,\infty) \times \mathbb{R}^n$ with the coordinates $(t,x) = (t, x_1 \ldots, x_n)$. Then there exists a constant C such that the following holds:

For every $G \in W^{k+1,p}(\mathbb{H}^{n+1})$ that is supported in $[0,a) \times V$ there exists $u \in W^{k+2,p}([0,a) \times V)$ such that

$$-\tfrac{\partial u}{\partial t}\big|_{t=0} = G|_{t=0} \qquad \textit{and} \qquad \|u\|_{W^{k+2,p}} \leq C\|G\|_{W^{k+1,p}}. \tag{3.6}$$

We will prove this by an explicit construction for u, and for that purpose we introduce the Poisson kernel on $\mathbb{R}^{n+1}$,

$$K(t,x) = K_t(x) = \begin{cases} \frac{1}{2\omega_2} \log(t^2 + x^2) & \text{if } n = 1, \\ -\frac{1}{(n-1)\omega_{n+1}}(t^2 + |x|^2)^{-\frac{n-1}{2}} & \text{if } n \geq 2. \end{cases}$$

Here $\omega_n = 2\pi^{\frac{n}{2}}\Gamma(\frac{n}{2})^{-1}$ is the volume of the unit sphere in $\mathbb{R}^n$. We restrict G to the boundary to obtain $G_0 := G|_{t=0} \in L^p(\mathbb{R}^n)$ (this is due to theorem B.10 and the fact that G has compact support). Now G_0 is integrable and has compact support, and K_t is smooth for $t > 0$, so we can define the function u on $(0,\infty) \times \mathbb{R}^n$ by convolution in $\mathbb{R}^n$,

$$u(t,x) := -(K_t * G_0)(x).$$

This function satisfies $\Delta u = 0$ since $\Delta K(t,x) = 0$ for all $(t,x) \neq 0$. Moreover, $-\frac{\partial u}{\partial t} = \sigma_t * G_0$ for $t > 0$, where

$$\sigma_t(x) \;:=\; \partial_t K(t,x) \;=\; \tfrac{t^{-n}}{\omega_{n+1}} \left(1 + \left|\tfrac{x}{t}\right|^2\right)^{-\frac{n+1}{2}}.$$

Note that $\sigma_t(x) = t^{-n}\beta(\frac{x}{t})$, where $\beta(x) = \omega_{n+1}^{-1}(1+|x|^2)^{-\frac{n+1}{2}}$ is a smooth function on $\mathbb{R}^n$ that satisfies

$$\int_{\mathbb{R}^n} \beta(x)\,\mathrm{d}^n x \;=\; \frac{\omega_n}{\omega_{n+1}} \int_{-\infty}^{\infty} \frac{r^{n-1}}{\sqrt{1+r^2}^{\,n+1}}\,\mathrm{d}r \;=\; 1.$$

Now the mollifier theorem C.5 asserts that $-\partial_t u|_{t=\varepsilon} = \sigma_\varepsilon * G_0$ converges to G_0 in the L^p-norm as $\varepsilon \to 0$. Once we have established that u is a $W^{k+2,p}$-function on $(0,a)\times V$ it is then clear that $-\partial_t u(0,x) = G(0,x)$ almost everywhere. This is due to the fact that $W^{1,p}$-functions restrict to L^p-functions on the boundary of compact domains, see theorem B.10. So it remains to prove the regularity of u and the corresponding estimate.

The Poisson kernel K is smooth for $t > 0$ and by assumption $G(t,y) = 0$ for all $t \geq a$ and $y \in \mathbb{R}^n$. Thus $u(t,x)$ can be rewritten as the sum of two convolutions in $\mathbb{R}^{n+1}$,

$$\begin{aligned}
u(t,x) &= -\int_{\mathbb{R}^n} K(t, x-y)\, G(0,y)\,\mathrm{d}^n y \\
&= \int_{\mathbb{R}^n}\int_0^\infty \tfrac{\partial}{\partial s}\left[K(t-s, x-y)\, G(s,y)\right]\,\mathrm{d}s\,\mathrm{d}^n y \\
&= -\int_{\mathbb{R}^n}\int_0^\infty \partial_t K(t-s, x-y)\, G(s,y)\,\mathrm{d}s\,\mathrm{d}^n y \\
&\quad + \int_{\mathbb{R}^n}\int_0^\infty K(t-s, x-y)\, \partial_t G(s,y)\,\mathrm{d}s\,\mathrm{d}^n y \\
&= -\partial_t K * \widetilde{G} + K * \widetilde{\partial_t G}.
\end{aligned}$$

Here for every function h on $\mathbb{H}^{n+1}$ we define an extended function on $\mathbb{R}^{n+1}$ (with coordinates (t,x)) by

$$\widetilde{h}(t,x) := \begin{cases} h(t,x) & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

The above expression can be used to define u on all of $\mathbb{R}^{n+1}$ since both K and $\partial_t K$ are integrable on compact sets and $\widetilde{G}, \widetilde{\partial_t G} \in L^p(\mathbb{R}^n)$ have compact support. Now let $J = (j_1, \ldots, j_\ell)$ with $1 \leq j_i \leq n$ and $\ell \leq k$ and denote by ∂_J a multiple derivative in the x-directions. Then $\partial_J \widetilde{G} = \widetilde{\partial_J G} \in L^p(\mathbb{R}^n)$ and $\partial_J \widetilde{\partial_t G} = \widetilde{\partial_J \partial_t G} \in L^p(\mathbb{R}^n)$, and hence

$$\partial_J u = -\partial_t K * \widetilde{\partial_J G} + K * \widetilde{\partial_J \partial_t G}. \tag{3.7}$$

For the second term on the right hand side one can use theorem C.4, which asserts that $K * \widetilde{\partial_J \partial_t G} \in W^{2,p}([0,a)\times V)$ and for some constant C

$$\|K * \widetilde{\partial_J \partial_t G}\|_{W^{2,p}([0,a)\times V)} \;\leq\; C \|\widetilde{\partial_J \partial_t G}\|_{L^p([0,a)\times V)} \;\leq\; C\|G\|_{W^{k+1,p}([0,a)\times V)}.$$

Theorem C.4 also provides a constant C such that the L^p-norm of the first term in (3.7) can be estimated by

$$\|\partial_t K * \widetilde{\partial_J G}\|_{L^p([0,a)\times V)} \;\leq\; C\|\widetilde{\partial_J G}\|_{L^p([0,a)\times V)} \;\leq\; C\|G\|_{W^{k,p}([0,a)\times V)}.$$

Moreover, the derivatives in x-direction of $\partial_t K * \widetilde{\partial_J G}$ can be expressed as

$$\partial_i\big(\partial_t K * \widetilde{\partial_J G}\big) = \partial_t K * \widetilde{\partial_i \partial_J G}$$

since $\partial_i \widetilde{\partial_J G} = \widetilde{\partial_i \partial_J G} \in L^p(\mathbb{R}^n)$ for $i = 1, \ldots, n$. Now again theorem C.4 yields a constant C such that

$$\big\|\partial_i\big(\partial_t K * \widetilde{\partial_J G}\big)\big\|_{L^p([0,a)\times V)} \;\leq\; C\|\widetilde{\partial_i \partial_J G}\|_{L^p([0,a)\times V)} \;\leq\; C\|G\|_{W^{k+1,p}([0,a)\times V)}.$$

Also, for $i = 1, \ldots, n$ and all derivatives ∂_α in t- or x-direction one has

$$\partial_\alpha \partial_i\big(\partial_t K * \widetilde{\partial_J G}\big) = \partial_\alpha \partial_t\big(K * \widetilde{\partial_i \partial_J G}\big).$$

Here the Calderon-Zygmund inequality (theorem C.3) can be applied since (by theorem C.4) we have $\Delta(K * \widetilde{\partial_i \partial_J G}) = \widetilde{\partial_i \partial_J G}$. Thus for some constant C

$$\begin{aligned}\big\|\partial_\alpha \partial_i\big(\partial_t K * \widetilde{\partial_J G}\big)\big\|_{L^p([0,a)\times V)} &\leq C\|\widetilde{\partial_i \partial_J G}\|_{L^p([0,a)\times V)} \\ &\leq C\|G\|_{W^{k+1,p}([0,a)\times V)}.\end{aligned}$$

So far we have proven that all derivatives of u up to order $k+2$ and including at most one derivative in t-direction lie in $L^p([0,a)\times V)$ and satisfy the corresponding estimate. For the derivatives including a higher number of t-derivatives this follows inductively from the fact that u is harmonic for $t > 0$. Thus we have on $(0,a)\times V$

$$\partial_t \partial_t u = -\partial_1 \partial_1 u - \ldots - \partial_n \partial_n u.$$

Assume that the regularity and estimate are established for derivatives including up to ℓ derivatives in t-direction (which was shown for $\ell = 1$). Now differentiate above equality by $\partial_t^{\ell-1}\partial_I$, where I is allowed to run through all multiple derivatives in x-direction of order at most $k-\ell+1$. Then by assumption all terms $\partial_t^{\ell-1}\partial_I \partial_i^2 u$ on the right hand side lie in $L^p([0,a)\times V)$ and satisfy the corresponding estimate, hence so does the left hand side. Thus we obtain regularity for up to $\ell+1$ derivatives in t-direction. This can be iterated until $\ell = k+2$. So it follows that in fact u restricted to $[0,a)\times V$ is a $W^{k+2,p}$-function satisfying (3.6), and this proves the claim.

2.) Proof of the theorem:

Now consider a compact manifold M with boundary ∂M. There exists a tubular neighbourhood diffeomorphism $\tau : [0,a)\times\partial M \xrightarrow{\sim} N \subset M$ such that $\tau|_{t=0} = \mathrm{Id}_{\partial M}$ and $\partial_t\tau|_{t=0} = -\nu$. (This can be constructed from the flow of a smooth vector field $\tilde{\nu}$ that equals $-\nu$ on ∂M.)

We construct coordinate charts for this neighbourhood N of ∂M as follows: Choose a finite atlas of the boundary, that is $\phi_i : V_i \to \mathbb{R}^n$ on $\partial M = \bigcup_{i=1}^K V_i$. Then an atlas of N is given by $N = \bigcup_{i=1}^K U_i$ with $U_i = \tau([0,a)\times V_i)$ and coordinate maps

$$\Phi_i : \begin{array}{ccc} U_i & \longrightarrow & [0,a)\times\phi_i(V_i) \subset \mathbb{H}^{n+1} \\ \tau(t,z) & \mapsto & (t,\phi_i(z)). \end{array}$$

Also choose a smooth partition of unity $1|_{\partial M} \equiv \sum_{i=1}^K \psi_i$, $\operatorname{supp}\psi_i \subset V_i$ and a smooth cutoff function $\rho : [0,a] \to [0,1]$ with $\rho \equiv 1$ near 0 and $\rho \equiv 0$ near a. Use these to define functions Ψ_i with support in U_i by $\Psi_i(\tau(t,z)) := \rho(t)\psi_i(z)$. These have the property that $\sum_{i=1}^K \Psi_i$ equals 1 in a neighbourhood of ∂M and that this sum can be smoothly extended by zero on $M \setminus N$.

Every given $G \in W^{1,p}(M)$ defines $W^{1,p}$-functions G_i on $\mathbb{H}^{n+1}$ with support in $[0,a)\times\phi_i(V_i)$,

$$G_i := \big(\Psi_i^{\frac{1}{2}} \cdot G\big)\circ\Phi_i^{-1}.$$

The local construction in 1.) now provides functions $u_i \in W^{2,p}([0,a)\times\phi_i(V_i))$ that satisfy (3.6) for $G = G_i$. These can be used to define the required function u:

$$u := \sum_{i=1}^K \Psi_i^{\frac{1}{2}} \cdot u_i\circ\Phi_i. \tag{3.8}$$

Here every summand $\Psi_i^{\frac{1}{2}} \cdot u_i\circ\Phi_i$ is extended by 0 on $M \setminus U_i$. This defines a $W^{2,p}$-function on M since both the cutoff function Ψ_i and the chart map Φ_i are smooth. Moreover, the estimate (3.6) between u_i and G_i implies

$$\begin{aligned} \|\Psi_i^{\frac{1}{2}} \cdot u_i\circ\Phi_i\|_{W^{2,p}(M)} &\le C_i\|u_i\|_{W^{2,p}([0,a)\times\phi_i(V_i))} \\ &\le C_i\big\|\big(\Psi_i^{\frac{1}{2}} \cdot G\big)\circ\Phi_i^{-1}\big\|_{W^{1,p}([0,a)\times\phi_i(V_i))} \\ &\le C'\|G\|_{W^{1,p}(M)}. \end{aligned}$$

Here C_i denotes any constant depending on i, but the final constant C' can be chosen such that it only depends on M and p. (For the equivalence of Sobolev norms on the manifold and in the charts see remark B.1.)

Now the regularity of the summands in (3.8) proves that $u \in W^{2,p}(M)$, and the above estimates for the summands can be summed up to get $\|u\|_{W^{2,p}} \leq C\|G\|_{W^{1,p}}$ with a finite constant C. It remains to check that u has the required normal derivative at the boundary: For $x \in \partial M$

$$\begin{aligned}
\tfrac{\partial u}{\partial \nu}(z) &= \mathrm{d}_z u(\nu) \\
&= -\tfrac{\partial}{\partial t}\big|_{t=0} u(\tau(t,z)) \\
&= -\tfrac{\partial}{\partial t}\Big|_{t=0} \sum_{i=1}^{K} \psi_i(z)^{\frac{1}{2}} \cdot \rho(t)^{\frac{1}{2}} \cdot u_i(t, \phi_i(z)) \\
&= \sum_{i=1}^{K} \psi_i(z)^{\frac{1}{2}} \cdot G_i(0, \phi_i(z)) \\
&= \sum_{i=1}^{K} \psi_i(z)^{\frac{1}{2}} \cdot \rho(0)^{\frac{1}{2}} \cdot \psi_i(z)^{\frac{1}{2}} \cdot G(z) \\
&= G(z).
\end{aligned}$$

Here we have used (3.6) and the facts that $\rho \equiv 1$ near $t = 0$ and $\sum_{i=1}^{K} \psi_i \equiv 1$ on ∂M. □

This construction can now be used to prove the sufficiency of the condition (3.2) for the existence of a solution of the Neumann problem (3.1) with inhomogeneous boundary conditions.

Proof of theorem 3.1 :
The remark just before the theorem shows the necessity of (3.2) for the existence of a solution of (3.1).

For the sufficiency let functions $f \in L^p(M)$ and $g \in W^{1,p}_{\partial}(M)$ be given that satisfy (3.2). Choose some $G \in W^{1,p}(M)$ with $G|_{\partial M} = g$ then by theorem 3.4 there exists $v \in W^{2,p}(M)$ that solves the boundary condition $\frac{\partial v}{\partial \nu} = G|_{\partial M} = g$. Now we have by assumption

$$\int_M (f - \Delta v) \;=\; \int_M f + \int_{\partial M} \tfrac{\partial v}{\partial \nu} \;=\; \int_M f + \int_{\partial M} g \;=\; 0.$$

Thus theorem 2.2 asserts the existence of a solution $\tilde{u} \in W^{2,p}(M)$ of the Neumann problem (NP) with f replaced by $f - \Delta v$. The solution of the inhomogeneous problem (3.1) is then given by $u = \tilde{u} + v \in W^{2,p}(M)$. Uniqueness follows from lemma 1.9. □

From the previous results we can deduce the Agmon-Douglis-Nirenberg estimate (3.3) and the more general regularity theorem 3.2.

Proof of theorem 3.2
Assume that $u \in \mathcal{D}(M)$ is a weak solution of the Neumann problem (3.4) for some given $f \in W^{k,p}(M)$ and $g \in W_\partial^{k+1,p}(M)$. Then (3.2) is satisfied automatically as follows from testing the weak equation with $\psi \equiv 1$. Thus by theorem 3.1 there exists a solution $v \in W^{k+2,p}(M)$ in the strong sense of the Neumann problem (3.1) for f and g. Then partial integration as in lemma N yields for all $\psi \in \mathcal{C}_\nu^\infty(M)$

$$\int_M (u-v)\Delta\psi \;=\; \int_M f\,\psi + \int_{\partial M} g\,\psi - \int_M \Delta v\,\psi - \int_{\partial M} \tfrac{\partial v}{\partial \nu}\,\psi \;=\; 0.$$

So $u - v$ is a weak solution of the homogeneous Neumann problem, hence by lemma 1.9 we have $u = v + C \in W^{k+2,p}(M)$ for some constant C.

To prove the estimates extend the unit normal vector field ν to a smooth vector field $\tilde{\nu}$ on M. Then we have $\frac{\partial u}{\partial \nu} = \mathrm{d}u(\tilde{\nu})|_{\partial M} \in W_\partial^{k+1,p}(M)$ for every $u \in W^{k+2,p}(M)$. Consider any $G \in W^{k+1,p}(M)$ such that $G|_{\partial M} = \frac{\partial u}{\partial \nu}$. By theorem 3.4 there exists $v \in W^{k+2,p}(M)$ such that

$$\tfrac{\partial v}{\partial \nu} = G|_{\partial M} = \tfrac{\partial u}{\partial \nu}, \qquad \|v\|_{W^{k+2,p}} \leq C\|G\|_{W^{k+1,p}}.$$

Now $u - v \in W^{k+2,p}(M)$ meets the homogeneous Neumann boundary condition, hence theorem 2.3 gives

$$\begin{aligned}
\|u\|_{W^{k+2,p}} &\leq \|u-v\|_{W^{k+2,p}} + \|v\|_{W^{k+2,p}} \\
&\leq C\big(\|\Delta u - \Delta v\|_{W^{k,p}} + \|u-v\|_{W^{k+1,p}}\big) + \|v\|_{W^{k+2,p}} \\
&\leq \tilde{C}\big(\|\Delta u\|_{W^{k,p}} + \|G\|_{W^{k+1,p}} + \|u\|_{W^{k+1,p}}\big).
\end{aligned}$$

This holds with the same constant for all $G \in W^{k+1,p}(M)$ that satisfy $G|_{\partial M} = \frac{\partial u}{\partial \nu}$, hence taking the infimum implies

$$\|u\|_{W^{k+2,p}} \leq \tilde{C}\left(\|\Delta u\|_{W^{k,p}} + \left\|\tfrac{\partial u}{\partial \nu}\right\|_{W_\partial^{k+1,p}} + \|u\|_{W^{k+1,p}}\right)$$

for every $u \in W^{k+2,p}(M)$. Finally, by lemma E.3 we can drop the term $\|u\|_{W^{k+1,p}}$ on the right hand side for functions with $\int_M u = 0$. This is due to the compactness of the Sobolev embedding $W^{k+2,p}(M) \hookrightarrow W^{k+1,p}(M)$ and the injectivity (see corollary 1.9) of the operator

$$\begin{array}{ccc}
\big\{u \in W^{k+2,p}(M) \;\big|\; \int_M u = 0\big\} & \longrightarrow & W^{k,p}(M) \times W_\partial^{k+1,p}(M) \\
u & \mapsto & (\Delta u, \frac{\partial u}{\partial \nu}).
\end{array}$$

The proof of the continuity of the constants claimed in remark 3.3 works analogously to the proof of theorem 5.1. So here we omit the technical details and only give the main arguments.

Firstly, the $W^{k+2,p}$-norm of a function u contains its L^p-norm and the $W^{k+1,p}$-norm of $\mathrm{d}u$. Now $\mathrm{d}u$ does not depend on the metric, and the covariant derivatives in the $W^{k+1,p}$-norm depend on the Christoffel symbols, i.e. the metric and its first derivatives. Hence $\|u\|_{W^{k+2,p}}$ depends continuously on the metric with respect to the $W^{k+1,\infty}$-topology. Next, $\Delta u = \mathrm{d}^*\mathrm{d}u$ contains the metric and its Christoffel symbols, hence $\|\Delta u\|_{W^{k,p}}$ also depends continuously on the metric with respect to the $W^{k+1,\infty}$-topology. The same holds for $\bigl\|\frac{\partial u}{\partial \nu}\bigr\|_{W_\partial^{k+1,p}}$ since for two different metrics g and h

$$\Bigl\|\frac{\partial u}{\partial \nu_g} - \frac{\partial u}{\partial \nu_h}\Bigr\|_{W_\partial^{k+1,p}} \;\leq\; \|\mathrm{d}u(\tilde{\nu}_g - \tilde{\nu}_h)\|_{W^{k+1,p}} \;\leq\; \|\tilde{\nu}_g - \tilde{\nu}_h\|_{W^{k+1,\infty}}\|u\|_{W^{k+2,p}}.$$

Here $\tilde{\nu}_g$ is some fixed vector field on M that extends the unit normal ν_g with respect to g and $\tilde{\nu}_h^i = h^{ij}g_{jk}\tilde{\nu}_g^k$ is defined such that it extends the unit normal ν_h with respect to h. Now

$$\|\tilde{\nu}_g - \tilde{\nu}_h\|_{W^{k+1,\infty}} \;\leq\; \|g^{-1} - h^{-1}\|_{W^{k+1,\infty}}\|g\|_{W^{k+1,\infty}}\|\tilde{\nu}_g\|_{W^{k+1,\infty}},$$

and by (E.4) this becomes arbitrarily small when h lies in a sufficiently small $W^{k+1,\infty}$-neighbourhood of g. These continuous dependencies prove the claimed continuity of C'.

For the continuity of C in the second estimate note that the condition $\int_M u = 0$ depends on the metric. So if $u \in W^{k+2,p}(M)$ meets it with respect to a metric h then the estimate with respect to the metric g only applies to $\sqrt{\det(g^{-1}h)}\cdot u$. Thus the derivatives of the metric occur up to order $k+2$ in the estimate and hence we need the metrics to be $W^{k+2,\infty}$-close for the continuity of the constant C. □

The results on the Neumann problem with inhomogeneous boundary conditions can also be summarized in the Fredholm property of the corresponding operator.

Corollary 3.5 *Let $k \in \mathbb{N}_0$ and $1 < p < \infty$. Then the operator of the Neumann problem with boundary conditions,*

$$D: \begin{array}{ccc} W^{k+2,p}(M) & \longrightarrow & W^{k,p}(M) \times W_\partial^{k+1,p}(M) \\ u & \mapsto & (\Delta u\,,\, \frac{\partial u}{\partial \nu}), \end{array}$$

is a Fredholm operator of index 0*.*

Proof: The estimate in corollary 3.2 shows that D has a finite dimensional kernel and a closed image, see lemma E.3 (i). In fact, by corollary 1.9 the kernel just consists of the constants. Theorem 3.1 asserts that the cokernel is isomorphic to the constants, too. Indeed, after adding a common constant every given pair (f, g) satisfies (3.2) and hence lies in the image of D. Thus the operator is Fredholm and its index is 0. □

Chapter 4

Sections of Vector Bundles

In this chapter we consider a generalized Neumann problem with inhomogenous boundary conditions for sections of vector bundles. In particular, we allow for the connections on the vector bundles to be not smooth but only of some lower Sobolev regularity. In principle one could directly deal with the Neumann problem in this generality since the leading terms are the same as in the scalar case. However, for simplicity we decided to treat this general situation seperately and only prove the results that will be needed for the local slice theorems F and F′. This does require an extension of the L^p-regularity result, and some more work arises from the lower order terms.

Let V be a finite dimensional vector space with an inner product and let M be a compact oriented Riemannian manifold with boundary ∂M and exterior unit normal vector field ν. In this chapter we consider a Riemannian vector bundle $V \hookrightarrow E \to M$ with fibre V over M that is equipped with a Riemannian covariant derivative

$$\nabla : \Gamma(E) \to \Gamma(\mathrm{T}^*M \otimes E).$$

Note that $\Gamma(\cdot)$ generally denotes the set of smooth sections of a bundle. Fix a bundle atlas (U_α, Φ_α) of E then sections $u \in \Gamma(E)$ and $\tau \in \Gamma(\mathrm{T}^*M \otimes E)$ are locally represented by maps $u_\alpha : U_\alpha \to V$ and 1-forms $\tau_\alpha \in \Omega^1(U_\alpha; V)$. The covariant derivative ∇ is locally represented by connection potentials $A_\alpha \in \Omega^1(U_\alpha; \mathrm{End}\, V)$ in the sense that

$$\nabla u_\alpha \;:=\; (\nabla u)_\alpha \;=\; \mathrm{d}u_\alpha + A_\alpha u_\alpha.$$

This evaluates as follows: For $X \in \mathrm{T}_pM$ with $p \in U_\alpha$

$$\nabla_X u \;:=\; (\nabla u)(X) \;=\; \mathrm{d}u_\alpha(X) + A_\alpha(X) u_\alpha(p),$$

where the endomorphism $A_\alpha(X)$ acts on $u_\alpha(p) \in V$. Next, the formally adjoint operator of ∇ is the coderivative

$$\nabla^* : \Gamma(\mathrm{T}^*M \otimes E) \to \Gamma(E)$$

defined by its local representation

$$\nabla^*\tau_\alpha \;:=\; (\nabla^*\tau)_\alpha \;:=\; \mathrm{d}^*\tau_\alpha + *(A_\alpha^* \wedge *\tau_\alpha).$$

This notation is to be understood as follows: The value of A_α^* on a vector in M is the adjoint (as an endomorphism of V) of the value of A^α on that vector and this endomorphism acts on the value of $*\tau_\alpha$ in V. One can check that this defines a global section of E and the subsequent lemma shows that it is in fact the formal adjoint of the covariant derivative.

However, due to a boundary term ∇^* is not actually dual to ∇. Thus for every $p \geq 1$ let $\frac{1}{p} + \frac{1}{p^*} = 1$ (that is $p^* = \infty$ in case $p = 1$) and denote by ∇' the dual operator of $\nabla : W^{1,p^*}(M,E) \to L^{p^*}(M, \mathrm{T}^*M \otimes E)$, i.e.

$$\nabla' :\; L^p(M, \mathrm{T}^*M \otimes E) \;\to\; W^{-1,p}(M,E) \;:=\; \big(W^{1,p^*}(M,E)\big)^*.$$

For $\tau \in L^p(M, \mathrm{T}^*M \otimes E)$ the linear form $\nabla'\tau$ acts on $u \in W^{1,p^*}(M,E)$ by

$$(\nabla'\tau)(u) = \int_M \langle\, \tau \wedge *\nabla u \,\rangle \,.$$

The notation $\langle \ldots \rangle$ indicates that the values of differential forms (in the fibres of E or in V) are combined with the corresponding inner product. Moreover, in the following functions on M or ∂M are always integrated with respect to the corresponding volume form (determined by the metric) – we suppress dvol_M and $\mathrm{dvol}_{\partial M}$.

Lemma 4.1 *Let $\tau \in \Gamma(\mathrm{T}^*M \otimes E)$ and $u \in \Gamma(E)$ then*

$$(\nabla'\tau)(u) \;=\; \int_M \langle\, \tau \wedge *\nabla u \,\rangle \;=\; \int_M \langle\, \nabla^*\tau \,,\, u \,\rangle + \int_{\partial M} \langle\, \tau(\nu) \,,\, u \,\rangle \,.$$

Proof: Choose a finite bundle atlas over $M = \bigcup_{\alpha=1}^N U_\alpha$ and a subordinate smooth partition of unity $1|_M \equiv \sum_{\alpha=1}^N \psi_\alpha$. Then on every U_α

$$\begin{aligned}
&\int_{U_\alpha} \psi_\alpha \,\langle\, \tau_\alpha \wedge *\nabla u_\alpha \,\rangle \\
&= \int_{U_\alpha} \psi_\alpha \,\langle\, \mathrm{d}u_\alpha \wedge *\tau_\alpha \,\rangle + \int_{U_\alpha} \psi_\alpha \,\langle\, A_\alpha u_\alpha \wedge *\tau_\alpha \,\rangle \\
&= \int_{U_\alpha} \psi_\alpha \,\langle\, u_\alpha \,,\, \mathrm{d}^*\tau_\alpha \,\rangle - \int_{U_\alpha} \mathrm{d}\psi_\alpha \wedge \langle\, u_\alpha * \tau_\alpha \,\rangle + \int_{\partial U_\alpha} \psi_\alpha \,\langle\, u_\alpha * \tau_\alpha \,\rangle \\
&\quad + \int_{U_\alpha} \psi_\alpha \,\langle\, u_\alpha \, A_\alpha^* \wedge *\tau_\alpha \,\rangle \\
&= \int_{U_\alpha} \psi_\alpha \,\langle\, \nabla^*\tau_\alpha \,,\, u_\alpha \,\rangle - \int_{U_\alpha} \mathrm{d}\psi_\alpha \wedge \langle\, u_\alpha * \tau_\alpha \,\rangle + \int_{\partial M \cap \partial U_\alpha} \psi_\alpha \,\langle\, u_\alpha * \tau_\alpha \,\rangle \,.
\end{aligned}$$

Note that the cutoff functions ψ_α vanish on the parts of ∂U_α that lie in the interior of M. Now we use the fact that the bundle is Riemannian, hence the values of the inner products below are independent of the choice of the trivialization. Hence we can use $\sum_{\alpha=1}^N \psi_\alpha = 1$ and $\sum_{\alpha=1}^N \mathrm{d}\psi_\alpha = 0$ to obtain

$$\begin{aligned}\int_M \langle\, \tau \wedge *\nabla u \,\rangle &= \sum_{\alpha=1}^N \int_{U_\alpha} \psi_\alpha \,\langle\, \tau_\alpha \wedge *(\nabla u)_\alpha \,\rangle \\ &= \sum_{\alpha=1}^N \biggl(\int_{U_\alpha} \psi_\alpha \,\langle\, (\nabla^* \tau)_\alpha \,,\, u_\alpha \,\rangle - \int_{U_\alpha} \mathrm{d}\psi_\alpha \wedge \langle\, u_\alpha * \tau_\alpha \,\rangle \\ &\qquad\qquad + \int_{\partial M \cap \partial U_\alpha} \psi_\alpha \,\langle\, u_\alpha * \tau_\alpha \,\rangle \biggr) \\ &= \int_M \langle\, \nabla^* \tau \,,\, u \,\rangle + \int_{\partial M} \langle\, u \,,\, \tau(\nu) \,\rangle \,.\end{aligned}$$

Here we also used the fact $*\tau|_{\partial M} = \tau(\nu)\,\mathrm{dvol}_{\partial M}$ from lemma 5.6 (i). □

This chapter deals with the following generalized Neumann boundary value problem for sections u of E :

$$\begin{cases} \nabla^* \nabla u = f & \text{on } M, \\ \nabla_\nu u = g & \text{on } \partial M. \end{cases} \tag{4.1}$$

Denote by $\mathcal{C}^\infty_\nu(M,E)$ the space of smooth sections $\psi \in \Gamma(E)$ with $\nabla_\nu \psi = 0$ on ∂M. Then a section $u \in L^1(M,E)$ is called a weak solution of (4.1) if

$$\int_M \langle\, u \,,\, \nabla^* \nabla \psi \,\rangle = \int_M \langle\, f \,,\, \psi \,\rangle + \int_{\partial M} \langle\, g \,,\, \psi \,\rangle \qquad \forall \psi \in \mathcal{C}^\infty_\nu(M,E). \tag{4.2}$$

If $u \in W^{1,p}(M,E)$ for some $p \geq 1$ and f and g are sufficiently regular then this weak equation can be written with the help of ∇',

$$\nabla' \nabla u = (f,g). \tag{4.3}$$

This is understood as follows: Every pair $(f,g) \in \mathcal{G}(E) \times \mathcal{G}(E|_{\partial M})$ defines a linear functional on $W^{1,p^*}(M,E)$ by

$$\psi \;\mapsto\; \int_M \langle\, f \,,\, \psi \,\rangle + \int_{\partial M} \langle\, g \,,\, \psi \,\rangle \,.$$

These strong and weak equations are equivalent for sufficiently regular sections as will be shown in the subsequent lemma. For that purpose we introduce spaces of $W^{k,p}$-sections restricted to the boundary analogous to the spaces $W^{k,p}_\partial(M)$ of real valued functions: For $k \in \mathbb{N}$ and $p \geq 1$ define

$$W^{k,p}_\partial(M,E) := \bigl\{ v|_{\partial M} \in \Gamma(E|_{\partial M}) \bigm| v \in W^{k,p}(M,E) \bigr\}.$$

As in the real valued case this becomes a Banach space when equipped with the norm $\|u\|_{W^{k,p}_\partial} = \inf\{ \|v\|_{W^{k,p}} \mid v \in W^{k,p}(M,E), v|_{\partial M} = u \}$.

Lemma 4.2 *Let $f \in L^p(M,E)$, $g \in W^{1,p}_\partial(M,E)$, and $u \in W^{2,p}(M,E)$ with $p \geq 1$. Then the equations (4.1), (4.2), and (4.3) for u are all equivalent.*

Proof: Let f, g, and u be given as supposed. Then lemma 4.1 provides (after approximation by smooth functions) for all $\psi \in W^{1,p^*}(M,E)$

$$\begin{aligned}(\nabla'\nabla u)(\psi) &= \int_M \langle \nabla u \wedge *\nabla\psi \rangle \\ &= \int_M \langle \nabla^*\nabla u \,,\, \psi \rangle + \int_{\partial M} \langle \nabla_\nu u \,,\, \psi \rangle \,.\end{aligned}$$

This shows that (4.1) implies (4.3). If in fact $\psi \in \mathcal{C}^\infty_\nu(M,E)$ then again by lemma 4.1

$$(\nabla'\nabla u)(\psi) \;=\; \int_M \langle \nabla u \wedge *\nabla\psi \rangle \;=\; \int_M \langle u \,,\, \nabla^*\nabla\psi \rangle$$

If u satisfies (4.3) then this immediately gives (4.2). It remains to show that (4.2) implies (4.1). So assume that u solves (4.2), then above calculations show that for all $\psi \in \mathcal{C}^\infty_\nu(M,E)$

$$\begin{aligned}\int_M \langle \nabla^*\nabla u \,,\, \psi \rangle + \int_{\partial M} \langle \nabla_\nu u \,,\, \psi \rangle &= (\nabla'\nabla u)(\psi) \\ &= \int_M \langle u \,,\, \nabla^*\nabla\psi \rangle \\ &= \int_M \langle f \,,\, \psi \rangle + \int_{\partial M} \langle g \,,\, \psi \rangle \,.\end{aligned}$$

From testing with $\psi \in \mathcal{C}^\infty_\nu(M,E)$ that have compact support in the interior of M one obtains $\nabla^*\nabla u = f$. Now locally in a trivialization over $U_\alpha \subset M$ and for all $\psi_\alpha \in \mathcal{C}^\infty(U_\alpha, V)$ that are supported in U_α and satisfy $\nabla_\nu\psi_\alpha = 0$ on $\partial M \cap U_\alpha$

$$\int_{\partial M\cap U_\alpha} \langle \nabla_\nu u_\alpha \,,\, \psi_\alpha \rangle = \int_{\partial M\cap U_\alpha} \langle g_\alpha \,,\, \psi_\alpha \rangle \,.$$

Here $\psi_\alpha|_{\partial M\cap U_\alpha}$ runs through all compactly supported smooth V-valued functions on $\partial M \cap U_\alpha$. Indeed, every such function ψ on $\partial M \cap U_\alpha$ can be smoothly extended to a function with support in U_α that has the prescribed normal derivative $\frac{\partial\psi}{\partial\nu} = -A_\alpha(\nu)\psi$, hence $\nabla_\nu\psi = 0$. So above identitiy implies $\nabla_\nu u_\alpha = g_\alpha$ in all bundle charts, and thus globally $\nabla_\nu u = g$. □

Here we have dealt with the following operators that are extensions of one another:

$$\begin{array}{rccc} \nabla'\nabla : & W^{1,p}(M,E) & \longrightarrow & W^{-1,p}(M,E) \\ & \cup & & \cup \\ \nabla^*\nabla \oplus \nabla_\nu : & W^{2,p}(M,E) & \longrightarrow & L^p(M,E) \oplus W^{1,p}_\partial(M,E) \\ & \cup & & \cup \\ \nabla^*\nabla : & W^{2,p}_\nu(M,E) & \longrightarrow & L^p(M,E) \end{array}$$

The inclusion $L^p(M,E)\oplus W^{1,p}_\partial(M,E)\subset W^{-1,p}(M,E)=\bigl(W^{1,p^*}(M,E)\bigr)^*$ is to be understood as explained for (4.3), just note that for $g=G|_{\partial M}\in W^{1,p}_\partial(M,E)$ and $\psi\in W^{1,p^*}(M,E)$ the function $\langle\, G\,,\,\psi\,\rangle$ lies in $W^{1,1}(M)$ and hence restricts to an L^1-function on ∂M that can be integrated (see lemma B.10).

Moreover, $W^{2,p}_\nu(M,E)$ denotes the space of $W^{2,p}$-sections u that satisfy the boundary condition $\nabla_\nu u=0$. On this space the Laplace type operator $\nabla^*\nabla$ coincides with $\nabla'\nabla$. However, for the treatment of inhomogenous boundary conditions one uses the intermediate operator $\nabla^*\nabla\oplus\nabla_\nu$.

Now let n denote the dimension of M, then the main results of this chapter – required for the local slice theorems F and F′ – are the following.

Theorem 4.3 *Let $2\le r<\infty$ such that $r>n$ and suppose that the connection potentials A_α are of class L^r. Then there exists a constant C such that for every $\tau\in L^r(M,\mathrm{T}^*M\otimes E)$ there exists a solution $u\in W^{1,r}(M,E)$ of*

$$\nabla'\nabla u=\nabla'\tau \qquad \textit{with} \qquad \|u\|_{W^{1,r}}\le C\|\nabla'\tau\|_{W^{-1,r}}.$$

Theorem 4.4 *Let $1<p\le q<\infty$ be such that*

$$p>\tfrac{n}{2} \qquad \textit{and} \qquad \tfrac{1}{n}>\tfrac{1}{q}\ge\tfrac{1}{p}-\tfrac{1}{n}.$$

Suppose that the connection potentials A_α are of class $W^{1,p}$. Then there exists a constant C such that for every $\tau\in W^{1,p}(M,\mathrm{T}^*M\otimes E)$ there exists a solution $u\in W^{2,p}(M,E)$ of

$$\begin{cases}\nabla^*\nabla u=\nabla^*\tau,\\ \quad\nabla_\nu u=\tau(\nu)\end{cases} \qquad \textit{with} \qquad \begin{aligned}&\|u\|_{W^{2,p}}\le C\bigl(\|\nabla^*\tau\|_p+\|\tau(\nu)\|_{W^{1,p}_\partial}\bigr),\\ &\|u\|_{W^{1,q}}\le C\|\tau\|_q.\end{aligned}$$

Remark 4.5 For later applications it will be necessary to vary the metric in the Neumann problem of theorem 4.4 (this affects ∇^* and ν). However, the constants in the estimates are not affected if the variation is small. More precisely, we will show the following:

In the situation of theorem 4.4 fix in addition a metric g on M. Then there exist constants $\varepsilon>0$ and C such that the assertion of the theorem holds for all metrics g' with $\|g-g'\|_{W^{1,\infty}}\le\varepsilon$. (Here we use g' in the boundary value problem but not for the Sobolev norms – these are equivalent anyway.)

In theorem 4.3 one considers the operator $\tilde D=\nabla'\nabla$ on $W^{1,r}(M,E)$ and theorem 4.4 concerns the operator $D=\nabla^*\nabla\oplus\nabla_\nu$ on $W^{2,p}(M,E)$. The operator D generalizes the operator $\Delta\oplus\frac{\partial}{\partial\nu}$ on $W^{2,p}(M)$ that is a Fredholm operator of index 0 as seen in corollary 3.5. The following two theorems generalize this Fredholm result to sections of vector bundles and to the weak problem (4.3), that is to the operators D and $\tilde D$. These theorems only concern the case of smooth connection potentials. The generalization to less regular covariant derivatives is part of the proof of theorem 4.3 and 4.4.

Theorem 4.6 *Let $1 < p < \infty$ and let ∇ be a covariant derivative on E with smooth connection potentials. Then*

$$D: \begin{array}{ccc} W^{2,p}(M,E) & \longrightarrow & L^p(M,E)\times W^{1,p}_\partial(M,E) \\ u & \mapsto & (\nabla^*\nabla u\,,\,\nabla_\nu u) \end{array}$$

is a Fredholm operator of index 0. Its kernel is the space of horizontal sections,

$$\ker D \;=\; \mathrm{H}^0(M,\nabla) \;:=\; \bigl\{u\in\Gamma(E)\bigm|\nabla u=0\bigr\}\subset\mathcal{C}^\infty(M,E),$$

the cokernel is isomorphic to $(\operatorname{im}D)^\perp = \bigl\{(v,v|_{\partial M})\bigm| v\in\mathrm{H}^0(M,\nabla)\bigr\}$,
and the image is $\operatorname{im}D = \bigl\{(\nabla^*\tau,\tau(\nu))\bigm|\tau\in W^{1,p}(M,\mathrm{T}^*M\otimes E)\bigr\}$.

Theorem 4.7 *Let $1 < q, q^* < \infty$ such that $\frac{1}{q^*}+\frac{1}{q}=1$ and let ∇ be a covariant derivative on E with smooth connection potentials, then*

$$\tilde{D}: \begin{array}{ccc} W^{1,q}(M,E) & \longrightarrow & W^{-1,q}(M,E) \\ u & \mapsto & \nabla'\nabla u \end{array}$$

is a Fredholm operator of index 0 with

$$\begin{aligned}\ker\tilde{D} &= \mathrm{H}^0(M,\nabla)\subset\mathcal{C}^\infty(M,E),\\ \operatorname{im}\tilde{D} &= \bigl\{\nabla'\tau\bigm|\tau\in L^q(M,\mathrm{T}^*M\otimes E)\bigr\}.\end{aligned}$$

Although the operator D locally has the same leading term as the operator of the Neumann problem in corollary 3.5, theorem 4.6 is not simply a corollary of the results on the Neumann problem. The difficulty lies in the lower order terms of the operator. Since there are also lower order terms in the operator ∇_ν on the boundary, one can not even use the construction of the Agmon-Douglis-Nirenberg theorem 3.4 to directly solve the inhomogenous boundary condition.

The hard part of theorem 4.6 is to verify the orthogonal complement $(\operatorname{im}D)^\perp$. This requires a regularity result on spaces $\bigl(W^{k,2}_\partial(M,E)\bigr)^*$ and a special trick to show that for $(v,\Phi)\in(\operatorname{im}D)^\perp$ the linear functional Φ on $W^{1,p}(M,E)$ satisfies $\Phi(u)=\int_{\partial M}\langle v\,,\,u\rangle$.

Proof of theorem 4.6 :
The proof consists of several steps: Firstly, we verify the kernel. Secondly, we prove that the image is closed, and then use this in III) to verify the cokernel. This proves the Fredholm property and index. Then in IV) we obtain the image explicitly.

I) First consider the kernel of D. We claimed that every $u\in W^{2,p}(M,E)$ with $Du=0$ is in fact smooth. Indeed, one sees inductively that its local representatives u_α lie in $W^{k,p}(U_\alpha,V)$ for all $k\geq 2$: If $u_\alpha\in W^{k,p}(U_\alpha,V)$ for some $k\geq 2$ then due to $Du=0$

$$\begin{aligned}\Delta u_\alpha &= -\mathrm{d}^*(A_\alpha u_\alpha) - *\bigl(A^*_\alpha\wedge *(\mathrm{d}u_\alpha + A_\alpha u_\alpha)\bigr),\\ \tfrac{\partial u_\alpha}{\partial\nu} &= -A_\alpha u_\alpha.\end{aligned}\tag{4.4}$$

These functions lie in $W^{k-1,p}(U_\alpha, V)$ and $W_\partial^{k,p}(U_\alpha, V)$ respectively. Thus the regularity theorem 3.2 implies that $u_\alpha \in W^{k+1,p}(U_\alpha, V)$. Hence all $u \in \ker D$ are smooth and we can apply lemma 4.1 to deduce $\nabla u = 0$:

$$\int_M \langle \nabla u \wedge *\nabla u \rangle \;=\; \int_M \langle u\,,\, \nabla^*\nabla u \rangle + \int_{\partial M} \langle u\,,\, \nabla_\nu u \rangle \;=\; 0.$$

This proves $\ker D = \mathrm{H}^0(M, \nabla)$. Now horizontal sections are uniquely determined on connected components by their value at one point. (This is since there is a well defined horizontal transport map along every smooth curve in M.) So $\mathrm{H}^0(M, \nabla)$ is finite dimensional since the dimension of V is finite and M has finitely many components.

II) Choose a finite bundle atlas over $M = \bigcup_{\alpha=1}^N U_\alpha$ and a subordinate smooth partition of unity $1|_M \equiv \sum_{\alpha=1}^N \psi_\alpha$. Then we can use theorem 3.2 in the local trivializations. Denote the local representatives of $u \in W^{2,p}(M, E)$ by u_α, then $\psi_\alpha u$ is represented by $\psi_\alpha u_\alpha$ over U_α and vanishes outside of U_α. Moreover, denote by ν_α the outward unit normal to ∂U_α. We have assumed that all derivatives of the connection potentials A_α are bounded, hence we obtain on every U_α

$$\begin{aligned}
\|\psi_\alpha u_\alpha\|_{W^{2,p}} &\le C\big(\|\Delta(\psi_\alpha u_\alpha)\|_p + \|\tfrac{\partial}{\partial \nu_\alpha}(\psi_\alpha u_\alpha)\|_{W_\partial^{1,p}} + \|\psi_\alpha u_\alpha\|_{W^{1,p}}\big) \\
&\le C_\alpha\big(\|\mathrm{d}^*\mathrm{d}u_\alpha\|_p + \|\psi_\alpha(\mathrm{d}u_\alpha)(\nu_\alpha)\|_{W_\partial^{1,p}} + \|u_\alpha\|_{W^{1,p}}\big) \\
&\le C_\alpha\big(\|\nabla^*\nabla u_\alpha\|_p + \| *(A_\alpha^* \wedge *\nabla u_\alpha)\|_p + \|\mathrm{d}^*(A_\alpha \wedge u_\alpha)\|_p \\
&\qquad + \|\psi_\alpha \nabla_{\nu_\alpha} u_\alpha\|_{W_\partial^{1,p}} + \|\psi_\alpha A_\alpha(\nu_\alpha) u_\alpha\|_{W_\partial^{1,p}} + \|u_\alpha\|_{W^{1,p}}\big) \\
&\le C_\alpha\big(\|\nabla^*\nabla u_\alpha\|_p + \|\psi_\alpha \nabla_{\nu_\alpha} u_\alpha\|_{W_\partial^{1,p}} + \|u_\alpha\|_{W^{1,p}}\big).
\end{aligned}$$

Here C_α denotes any constant depending on A_α and ψ_α.
Summing this up yields

$$\begin{aligned}
\|u\|_{W^{2,p}} &\le \sum_{\alpha=1}^N \|\psi_\alpha u_\alpha\|_{W^{2,p}(U_\alpha)} \\
&\le \sum_{\alpha=1}^N C_\alpha \Big(\|\nabla^*\nabla u_\alpha\|_{L^p(U_\alpha)} + \|\psi_\alpha \nabla_{\nu_\alpha} u_\alpha\|_{W_\partial^{1,p}(U_\alpha)} + \|u_\alpha\|_{W^{1,p}(U_\alpha)}\Big) \\
&\le C\big(\|\nabla^*\nabla u\|_p + \|\nabla_\nu u\|_{W_\partial^{1,p}} + \|u\|_{W^{1,p}}\big).
\end{aligned}$$

The $W_\partial^{1,p}$-norm on the boundary of U_α can be replaced by the norm on the boundary of M since the ψ_α vanish on ∂U_α except for where it intersects ∂M. By lemma E.3 (i) and the compactness of the Sobolev embedding $W^{2,p} \hookrightarrow W^{1,p}$ this inequality asserts that the image of D is closed.

III) To establish the Fredholm property and the index 0 it remains to show that the codimension of the image of D is equal to the dimension of the kernel

$\mathrm{H}^0(M,\nabla)$. The cokernel $\big(L^p(M,E)\times W^{1,p}_\partial(M,E)\big)/\operatorname{im} D$ is is a Banach space since it is the quotient of a Banach space by a closed subspace. Its dimension is the same as the dimension of its dual space, and this is isomorphic to the annihilator $(\operatorname{im} D)^\perp \subset \big(L^p(M,E)\times W^{1,p}_\partial(M,E)\big)^*$ of the image. So consider $(v,\Phi)\subset(\operatorname{im} D)^\perp$, that is $v\in L^{p^*}(M,E)$ and $\Phi\in\big(W^{1,p}_\partial(M,E)\big)^*$ such that

$$\int_M \langle\, v\,,\, \nabla^*\nabla u\,\rangle + \Phi(\nabla_\nu u) = 0 \qquad \forall u\in W^{2,p}(M,E). \tag{4.5}$$

Our aim is to prove that $v\in\mathrm{H}^0(E,\nabla)$ and that $\Phi = v|_{\partial M}$ in the sense that $\Phi(u)=\int_{\partial M}\langle\, v\,,\, u\,\rangle$ for all $u\in W^{1,p}_\partial(M,E)$. The proceeding of the proof is to first obtain better regularity of v in order to be able to perform partial integration in the first term of (4.5). This will then allow to deduce $\Phi = v|_{\partial M}$, and the remaining equality with $u=v$ will assert $\nabla v = 0$.

So firstly, one obtains $v\in W^{1,2}(M,E)$ and $\Phi\in\big(W^{1,2}_\partial(M,E)\big)^*$ from iteration of the following claim.

<u>Claim:</u> *Suppose $v\in W^{-k,2}(M,E)$ and $\Phi\in\big(W^{k+1,2}_\partial(M,E)\big)^*$, then it follows from (4.5) that in fact $v\in W^{-k+1,2}(M,E)$, and as long as $k\geq 1$ one also obtains $\Phi\in\big(W^{k,2}_\partial(M,E)\big)^*$.*

To establish this claim choose a finite bundle atlas over $M=\bigcup_{\alpha=1}^N U_\alpha$ and find subsets $W_\alpha\subset U_\alpha$ that still cover M, but such that $\overline{W_\alpha}$ is disjoint from $U_\alpha^{\complement}$. Then there exist cutoff functions $\psi_\alpha\in\mathcal{C}^\infty(M)$ that are supported in U_α and equal 1 on W_α. For all smooth sections $u\in\Gamma(E)$ we apply (4.5) to $\psi_\alpha u$ instead of u and obtain

$$\begin{aligned}
0 &= \int_M \langle\, v\,,\, \nabla^*\nabla(\psi_\alpha u)\,\rangle + \Phi\big(\nabla_\nu(\psi_\alpha u)\big)\\
&= \int_{U_\alpha}\langle\, \psi_\alpha v_\alpha\,,\, \Delta u_\alpha\,\rangle - \int_{U_\alpha}\langle\, v_\alpha\,,\, R(u_\alpha)\,\rangle + \Phi\big(\tfrac{\partial\psi_\alpha}{\partial\nu}u_a + \psi_\alpha A_\alpha(\nu)u_\alpha + \psi_\alpha\tfrac{\partial u_\alpha}{\partial\nu}\big)
\end{aligned}$$

with

$$R(\phi) = (\Delta-\nabla^*\nabla)(\psi_\alpha\phi) + 2\,\langle\, \mathrm{d}\psi_\alpha\,,\, \mathrm{d}\phi\,\rangle - \Delta\psi_\alpha\cdot\phi.$$

For all $\phi\in\mathcal{C}^\infty(U_\alpha,V)$ with $\frac{\partial\phi}{\partial\nu}|_{\partial U_\alpha}=0$ above equality with $u_\alpha=\phi$ yields

$$\begin{aligned}
\left|\int_{U_\alpha}\langle\, \psi_\alpha v_\alpha\,,\, \Delta\phi\,\rangle\right| &\leq \left|\int_{U_\alpha}\langle\, v_\alpha\,,\, R(\phi)\,\rangle\right| + \left|\Phi\big(\tfrac{\partial\psi_\alpha}{\partial\nu}\phi + \psi_\alpha A_\alpha(\nu)\phi\big)\right|\\
&\leq \|v_\alpha\|_{W^{-k,2}}\|R(\phi)\|_{W^{k,2}} + C\|\Phi\|_{(W^{k+1,2}_\partial)^*}\|\phi\|_{W^{k+1,2}_\partial}\\
&\leq C'\|\phi\|_{W^{k+1,2}}.
\end{aligned}$$

Here the constant C depends on A_α and ψ_α, and C' depends in addition on v and Φ. This asserts that $\psi_\alpha v_\alpha$ solves the weak (V-valued) Neumann problem (wNP) for some $f\in W^{-k-1,2}(U_\alpha,V)$. Hence $\psi_\alpha v_\alpha\in W^{-k+1,2}(U_\alpha,V)$ by the

L^2-regularity theorem 1.3. Then $v_\alpha \in W^{-k+1,2}(W_\alpha, V)$ for all α and hence (see remark B.1) $v \in W^{-k+1,2}(M, E)$.

This can be used to improve the regularity of Φ in case $k \geq 1$. For that purpose fix a partition of unity $1|_M \equiv \sum_{\alpha=1}^N \psi_\alpha$ with $\operatorname{supp}\psi_\alpha \subset U_\alpha$ and assume the bundle charts to have smooth boundaries ∂U_α. Consider any section $g \in \Gamma(E|_{\partial M})$ and choose some extension $G \in \Gamma(E)$ such that $G|_{\partial M} = g$ and $\|G\|_{W^{k,2}} \leq 2\|g\|_{W_\partial^{k,2}}$. [1] Then by theorem 3.4 one finds $u_\alpha \in W^{k+1,2}(U_\alpha, V)$ for all α such that $\frac{\partial u_\alpha}{\partial \nu_\alpha} = G_\alpha$ on ∂U_α, in particular $\frac{\partial u_\alpha}{\partial \nu} = g_\alpha$ on $\partial M \cap U_\alpha$, and for some constant C_α

$$\|u_\alpha\|_{W^{k+1,2}} \;\leq\; C_\alpha \|G_\alpha\|_{W^{k,2}} \;\leq\; 2C_\alpha \|g\|_{W_\partial^{k,2}}.$$

Now apply (4.5) to every $\psi_\alpha u_\alpha$ and obtain

$$\begin{aligned}
|\Phi(g)| &\leq \sum_{\alpha=1}^N |\Phi(\psi_\alpha g)| \\
&\leq \sum_{\alpha=1}^N \Big(\big|\Phi\big(\nabla_\nu(\psi_\alpha u_\alpha)\big)\big| + \big|\Phi\big(\tfrac{\partial \psi_\alpha}{\partial \nu} u_\alpha + \psi_\alpha A_\alpha(\nu) u_\alpha\big)\big| \Big) \\
&\leq \sum_{\alpha=1}^N \left(\left| \int_{U_\alpha} \langle\, v_\alpha \,,\, \nabla^* \nabla(\psi_\alpha u_\alpha) \,\rangle \right| + C_\alpha \|\Phi\|_{(W_\partial^{k+1,2})^*} \|u_\alpha\|_{W_\partial^{k+1,2}} \right) \\
&\leq \sum_{\alpha=1}^N C_\alpha \left(\|v_\alpha\|_{W^{-k+1,2}} + \|\Phi\|_{(W_\partial^{k+1,2})^*} \right) \|u_\alpha\|_{W^{k+1,2}} \\
&\leq C\|g\|_{W_\partial^{k,2}}.
\end{aligned}$$

Here C_α denotes any constant depending on α and the constant C depends on v and Φ. This holds for all $g \in \Gamma(E|_{\partial M})$ and hence assserts that $\Phi \in \big(W_\partial^{k,2}(M,E)\big)^*$, which proves the claim.

Remember that we consider $v \in L^{p^*}(M,E)$ and $\Phi \in \big(W_\partial^{1,p}(M,E)\big)^*$ that satisfy (4.5). These meet the assumptions of above claim for some integer $k \geq \frac{n}{2}$ due to the dual of the Sobolev embedding $W^{k,2} \hookrightarrow L^p$. So iteration of the claim proves that $v \in L^2(M,E)$ and $\Phi \in \big(W_\partial^{1,2}(M,E)\big)^*$. Then the first part of the claim moreover implies that $v \in W^{1,2}(M,E)$. Now one can use lemma 4.1 in (4.5)

[1] To see that an extension exists one uses a tubular neighbourhood $U \cong \partial M \times [0,a)$ of ∂M: Extend the section from ∂M to U by horizontal transport along the $[0,a)$-curves and multiply it with a smooth cutoff function with respect to the $[0,a)$-coordinate. This section can then be smoothly extended by 0 to the rest of M.

One moreover finds an extension whose $W^{k,2}$-norm is arbitrarily close to the $W_\partial^{k,2}$-norm of the given section on the boundary. (In the first place, this section will not be smooth, but one can choose a nearby smooth section.) This gives the required inequality unless the $W_\partial^{k,2}$-norm vanishes. If $g \equiv 0$ then one simply chooses the extension $G \equiv 0$.

to obtain for all $u \in \Gamma(E)$

$$\begin{aligned} 0 &= \int_M \langle\, v\,,\, \nabla^*\nabla u\,\rangle + \Phi(\nabla_\nu u) \\ &= \int_M \langle\, \nabla v \wedge *\nabla u\,\rangle - \int_{\partial M} \langle\, v\,,\, \nabla_\nu u\,\rangle + \Phi(\nabla_\nu u). \end{aligned} \tag{4.6}$$

This will now be used to prove $\Phi = v|_{\partial M}$, that is $\Phi(g) = \int_{\partial M} \langle\, v\,,\, g\,\rangle$ for every $g \in \Gamma(E|_{\partial M})$. Actually, by the use of a partition of unity it suffices to establish this equality in the case when g is supported in some $\partial M \cap U_\alpha$.

Ideally one would like to find $u \in \Gamma(E)$ with $\nabla_\nu u = g$ such that (4.6) gives $\left|\Phi(g) - \int_{\partial M} \langle\, v\,,\, g\,\rangle\right| \leq \|\nabla v\|_2 \|\nabla u\|_2$. If now ∇u was arbitrarily small in the L^2-norm then that would show $\Phi = v|_{\partial M}$. However, the lower order terms in ∇_ν provide some difficulties, so we only solve $\frac{\partial u_\alpha}{\partial \nu} = g_\alpha$. More precisely, given any $g \in \Gamma(E|_{\partial M})$ that is supported in $\partial M \cap U_\alpha$ (and hence is represented by $g_\alpha \in \mathcal{C}^\infty(\partial M \cap U_\alpha, V)$) we will construct $u \in \Gamma(E)$ supported in U_α such that $\|u\|_{W^{1,2}}$ is arbitrarily small and $\frac{\partial u_\alpha}{\partial \nu} = g_\alpha$ holds on $\partial M \cap U_\alpha$:

There exists a tubular neighbourhood $\tau : \operatorname{supp} g \times [0, a) \to U_\alpha$ such that $\frac{\partial}{\partial t}\big|_{t=0} \tau(z,t) = -\nu(z)$ for all $z \in \operatorname{supp} g$, where t is the $[0,a)$-coordinate. Next, for every $a > \varepsilon > 0$ there is a smooth cutoff function $\phi : [0, a) \to [0, \frac{\varepsilon}{2}]$ as in figure 4.1 that satisfies $\phi'(0) = -1$, $\phi(t) = 0$ for $t \geq \varepsilon$, and $|\phi'| \leq 1$. We also restrict to $\varepsilon \leq 2$ such that $|\phi| \leq 1$.

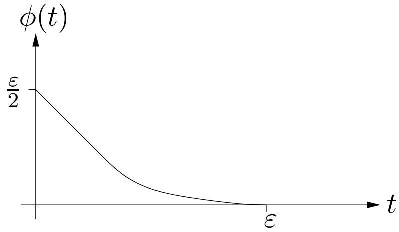

Figure 4.1: Cutoff function ϕ

We define $u \in \Gamma(E)$ by $u_\alpha(\tau(z,t)) := \phi(t) g_\alpha(z)$, which extends smoothly by 0 to all of M. Then indeed

$$\tfrac{\partial u_\alpha}{\partial \nu}(z) \;=\; -\tfrac{\partial}{\partial t}\big|_{t=0} u_\alpha(\tau(z,t)) \;=\; -\phi'(0) g_\alpha(z) \;=\; g_\alpha(z).$$

Moreover, all derivatives of τ^{-1} and g_α are bounded and also ϕ and ϕ' are bounded independently of ε. Thus u_α and its first derivatives are bounded independently of ε and its support is contained in $\tau(\operatorname{supp} g \times [0, \varepsilon])$. Hence for some finite constants C and C'

$$\|u\|_{W^{1,2}} \;\leq\; C' \operatorname{Vol}(\tau(\operatorname{supp} g \times [0,\varepsilon]))^{\frac{1}{2}} \;\leq\; C\varepsilon^{\frac{1}{2}}.$$

Here we used the fact that all derivatives of τ are bounded, hence the volume in M can be bounded above by some multiple of the volume in $\operatorname{supp} g \times [0, a)$. So by the choice of $\varepsilon > 0$ one can make $\|u\|_{W^{1,2}}$ arbitrarily small.

Now for every $g \in \Gamma(E|_{\partial M})$ let $u \in \Gamma(E)$ be constructed as above, then $\nabla_\nu u - g$ is a section of $E|_{\partial M}$ that is supported in $\partial M \cap U_\alpha$ where it is represented by

$A_\alpha(\nu)u_\alpha \in \mathcal{C}^\infty(\partial M \cap U_\alpha, V)$. Insert this section u in (4.6) to obtain

$$\begin{aligned}
&\left|\Phi(g) - \int_{\partial M} \langle\, v\,,\, g\,\rangle\right| \\
&\leq \left|\Phi(\nabla_\nu u) - \int_{\partial M} \langle\, v\,,\, \nabla_\nu u\,\rangle\right| + \left|\int_{\partial M} \langle\, v\,,\, \nabla_\nu u - g\,\rangle\right| + \left|\Phi\big(\nabla_\nu u - g\big)\right| \\
&\leq \left|\int_M \langle\, \nabla v \wedge *\nabla u\,\rangle\right| + \|v\|_{L^2(\partial M)}\|\nabla_\nu u - g\|_{L^2(\partial M)} + \|\Phi\|_{(W^{1,2}_\partial)^*}\|\nabla_\nu u - g\|_{W^{1,2}_\partial} \\
&\leq \|\nabla v\|_2\|\nabla u\|_2 + C\left(\|v\|_{W^{1,2}} + \|\Phi\|_{(W^{1,2}_\partial)^*}\right)\|A_\alpha(\tilde{\nu})u_\alpha\|_{W^{1,2}(U_\alpha)} \\
&\leq C\left(\|v\|_{W^{1,2}} + \|\Phi\|_{(W^{1,2}_\partial)^*}\right)\|u\|_{W^{1,2}}.
\end{aligned}$$

Here C denotes any finite constant and $\tilde{\nu} \in \Gamma(\mathrm{T}M)$ is an extension of ν. Moreover, we have used the trace theorem B.10 to estimate the L^2-norms on ∂M by the $W^{1,2}$-norms on M.

Now $\|u\|_{W^{1,2}}$ can be chosen arbitrarily small, so above inequality shows that indeed $\Phi(g) = \int_{\partial M} \langle\, v\,,\, g\,\rangle$ for all considered sections $g \in \Gamma(E|_{\partial M})$, and hence $\Phi = v|_{\partial M}$. Use this in (4.6) to obtain for all $u \in \Gamma(E)$

$$\int_M \langle\, \nabla v \wedge *\nabla u\,\rangle = 0.$$

Since $v \in W^{1,2}(M,E)$ this identity extends to all $u \in W^{1,2}(M,E)$ and thus also holds for $u = v$. This proves $\nabla v = 0$. The regularity $v \in \mathcal{C}^\infty(M,E)$ follows by induction: Suppose $v \in W^{k,2}(M,E)$ (we have established that for $k = 1$) then in all charts $\mathrm{d}v_\alpha = -A_\alpha v_\alpha \in W^{k,2}(U_\alpha, \mathrm{T}^*U_\alpha \otimes V)$ and hence actually $v \in W^{k+1,2}(M,E)$. So we have shown

$$(\operatorname{im} D)^\perp \;=\; \big\{(v, v|_{\partial M} \,\big|\, v \in \mathrm{H}^0(M,\nabla)\big\} \;\cong\; \mathrm{H}^0(M,\nabla).$$

This proves that the cokernel of D is of finite dimension equal to the dimension of the kernel, hence D is Fredholm and its index is 0.

IV) Consider

$$Z := \big\{\big(\nabla^*\tau, \tau(\nu)\big) \,\big|\, \tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E)\big\}.$$

Note that we have $Z \subset (\operatorname{im} D)^{\perp\perp} = \overline{\operatorname{im} D} = \operatorname{im} D$ since by lemma 4.1 for all $v \in \mathrm{H}^0(M,\nabla)$ and $\tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E)$

$$\int_M \langle\, v\,,\, \nabla^*\tau\,\rangle + \int_{\partial M} \langle\, v\,,\, \tau(\nu)\,\rangle \;=\; \int_M \langle\, \nabla v \wedge *\tau\,\rangle \;=\; 0.$$

But $\operatorname{im} D \subset Z$ also holds since ∇ maps $W^{2,p}(M,E)$ to $W^{1,p}(M, \mathrm{T}^*M \otimes E)$. Hence Z is indeed the image of D. □

One can use the results in theorem 4.6 to determine $\ker \tilde{D}$ and $(\operatorname{im} \tilde{D})^{\perp}$ for theorem 4.7. The only hard part of the proof is then the closedness of $\operatorname{im} \tilde{D}$ that will be proven by an estimate.

Proof of theorem 4.7 :
Let $u \in \ker \tilde{D}$, then lemma 4.1 asserts for all $\psi \in W^{2,q^*}(M,E)$

$$0 \;=\; \nabla'\nabla u(\psi) \;=\; \int_M \langle\, \nabla u \wedge *\nabla\psi \,\rangle \;=\; \int_M \langle\, u\,,\, \nabla^*\nabla\psi \,\rangle + \int_{\partial M} \langle\, u\,,\, \nabla_\nu\psi \,\rangle\,.$$

Thus $(u, u|_{\partial M}) \in (\operatorname{im} D)^{\perp}$ with the operator D of theorem 4.6 for $p = q^*$, and this implies that $u \in \mathrm{H}^0(M,\nabla)$. On the other hand every horizontal section obviously lies in the kernel of $\tilde{D}$, so $\ker \tilde{D} = \mathrm{H}^0(M,\nabla)$ and this is of finite dimension as before in theorem 4.6.

The same argument can be used to show that $(\operatorname{im} \tilde{D})^{\perp} = \mathrm{H}^0(M,\nabla)$: Let $u \in (\operatorname{im} \tilde{D})^{\perp} \subset W^{1,q^*}(M,E)$, i.e. $\tilde{D}\psi(u) = 0$ for all $\psi \in W^{1,q}(M,E)$. Then for all $\psi \in W^{2,q}(M,E)$ by lemma 4.1

$$0 \;=\; \nabla'\nabla\psi(u) \;=\; \int_M \langle\, \nabla\psi \wedge *\nabla u \,\rangle \;=\; \int_M \langle\, u\,,\, \nabla^*\nabla\psi \,\rangle + \int_{\partial M} \langle\, u\,,\, \nabla_\nu\psi \,\rangle\,.$$

This shows $(u, u|_{\partial M}) \in (\operatorname{im} D)^{\perp}$ with $p = q$, and thus theorem 4.6 asserts that $u \in \mathrm{H}^0(M,\nabla)$. Conversely, every $u \in \mathrm{H}^0(M,\nabla)$ satisfies

$$\tilde{D}\psi(u) \;=\; \int_M \langle\, \nabla\psi \wedge *\nabla u \,\rangle \;=\; 0$$

for all $\psi \in W^{1,q}(M,E)$. So we have established $(\operatorname{im} \tilde{D})^{\perp} = \mathrm{H}^0(M,\nabla)$.

If we can moreover show that $\operatorname{im} \tilde{D}$ is closed then the quotient norm is welldefined on the cokernel $W^{1,q}(M,E)/\operatorname{im} \tilde{D}$ and makes it a Banach space. The cokernel has the same dimension as its dual space, which is isomorphic to $(\operatorname{im} \tilde{D})^{\perp}$. Thus $\operatorname{codim} \operatorname{im} \tilde{D} = \dim \mathrm{H}^0(M,\nabla) = \dim \ker \tilde{D}$ proving the Fredholm property and index 0 of $\tilde{D}$.

The closedness of $\operatorname{im} \tilde{D}$ will follow from lemma E.3 (i). This requires an estimate that uses some topological direct sums. In the following we indicate by a subscript the spaces that ∇ and ∇' act on, e.g.

$$\begin{aligned} \nabla'_{q^*} :&\quad L^{q^*}(M, \mathrm{T}^*M \otimes E) \;\longrightarrow\; W^{-1,q}(M,E), \\ \nabla_{q^*} :&\quad W^{1,q^*}(M,E) \;\longrightarrow\; L^{q^*}(M, \mathrm{T}^*M \otimes E). \end{aligned}$$

Firstly, $\ker \nabla_{q^*} = \mathrm{H}^0(M,E)$ is finite dimensional, so (e.g. by [Z, 3.9,Ex.17]) it has a topological (i.e. closed) complement $Y \subset W^{1,q^*}(M,E)$,

$$W^{1,q^*}(M,E) = \mathrm{H}^0(M,E) \oplus Y. \tag{4.7}$$

If we restrict ∇ to this complement then it becomes injective. Moreover, we have $\|u\|_{W^{1,q^*}} \le \|\nabla u\|_{q^*} + \|u\|_{q^*}$ for all $u \in W^{1,q^*}(M,E)$, and the Sobolev embedding $W^{1,q^*} \hookrightarrow L^{q^*}$ is compact. So $\operatorname{im}\nabla_{q^*} \subset L^{q^*}(M, \mathrm{T}^*M \otimes E)$ is closed by lemma E.3 (i), and then (ii) implies that there exists a constant C such that

$$\|u\|_{W^{1,q^*}} \le C\|\nabla u\|_{q^*} \qquad \forall u \in Y. \tag{4.8}$$

Secondly, we claim the following topological direct sum:

$$L^{q^*}(M, \mathrm{T}^*M \otimes E) = \ker \nabla'_{q^*} \oplus \operatorname{im}\nabla_{q^*}. \tag{4.9}$$

Both factors are closed subsets of $L^{q^*}(M, \mathrm{T}^*M \otimes E)$ (for $\operatorname{im}\nabla_{q^*}$ this was seen above) and the intersection $\ker\nabla'_{q^*} \cap \operatorname{im}\nabla_{q^*}$ is trivial. Indeed, let τ lie in the intersection, that is $\tau = \nabla v$ for some $v \in W^{1,q^*}(M,E)$ such that $\tilde{D}v = \nabla'\tau = 0$. Then $v \in \ker\tilde{D} = \mathrm{H}^0(M,\nabla)$ (as was already established with q replaced by q^*) and hence $\tau = \nabla v = 0$.

In order to see that the closed subspace $X := \ker\nabla'_{q^*} \oplus \operatorname{im}\nabla_{q^*}$ is in fact the whole space $L^{q^*}(M, \mathrm{T}^*M \otimes E)$ assume the contrary, i.e. that there exists an element in $L^{q^*}(M, \mathrm{T}^*M \otimes E)$ that has nonzero distance from X. The theorem of Hahn-Banach (e.g. [Z, 1.2,Prop.3]) then provides a nonzero $\omega \in L^q(M, \mathrm{T}^*M \otimes E)$ such that $\omega \in X^\perp = (\ker\nabla'_{q^*})^\perp \cap (\operatorname{im}\nabla_{q^*})^\perp = \operatorname{im}\nabla_q \cap \ker\nabla'_q$. But as before this intersection is trivial, so we arrived at a contradiction. Here $(\operatorname{im}\nabla_{q^*})^\perp = \ker\nabla'_q$ follows directly from the definition of ∇'. This also holds with q and q^* interchanged, and the closedness of $\operatorname{im}\nabla_q$ then provides $(\ker\nabla'_{q^*})^\perp = (\operatorname{im}\nabla_q)^{\perp\perp} = \operatorname{im}\nabla_q$.

We now want to use (4.7) - (4.9) to estimate ∇u by $\tilde{D}u$, which is the crucial estimate for the closedness of $\operatorname{im}\tilde{D}$. For that purpose we use the duality of L^q and L^{q^*} to write

$$\|\nabla u\|_q = \sup\left\{ \frac{\left|\int_M \langle\, \nabla u \wedge *\tau \,\rangle\right|}{\|\tau\|_{q^*}} \;\middle|\; \tau \in L^{q^*}(M, \mathrm{T}^*M \otimes E) \right\}. \tag{4.10}$$

By (4.9) one can write all $\tau \in L^{q^*}(M, \mathrm{T}^*M \otimes E)$ in the form $\tau = \nabla v + \omega$ for some $v \in W^{1,q^*}(M,E)$ and $\omega \in \ker\nabla'_{q^*}$. This splitting is continuous, so $\|\nabla v\|_{q^*} \le C'\|\tau\|_{q^*}$ for some constant C'. Due to (4.7) one can moreover choose $v \in Y$, then (4.8) provides a constant C such that

$$C\|\tau\|_{q^*} \ge \|v\|_{W^{1,q^*}}.$$

Moreover, the splitting $\tau = \nabla v + \omega$ gives

$$\int_M \langle\, \nabla u \wedge *\tau \,\rangle \;=\; (\nabla'\nabla u)(v) + (\nabla'\omega)(u) \;=\; (\tilde{D}u)(v).$$

Now insert this into (4.10), then for all $u \in W^{1,q}(M,E)$

$$\|\nabla u\|_q \;\le\; \sup\left\{ \frac{|\tilde{D}u(v)|}{\frac{1}{C}\|v\|_{W^{1,q^*}}} \;\middle|\; v \in Y \right\} \;\le\; C\|\tilde{D}u\|_{W^{-1,q}}$$

and hence

$$\|u\|_{W^{1,q}} \;\le\; \|\nabla u\|_q + \|u\|_q \;\le\; C\|\tilde{D}u\|_{W^{-1,q}} + \|u\|_q .$$

The embedding $W^{1,q} \hookrightarrow L^q$ is compact, so by lemma E.3 (i) this estimate implies that $\operatorname{im}\tilde{D}$ is closed.

To determine the image of $\tilde{D}$ explicitly note that $\operatorname{im}\tilde{D} \subset \operatorname{im}\nabla'_q$ since ∇ maps $W^{1,q}(M,E)$ to $L^q(M,\mathrm{T}^*M\otimes E)$. On the other hand, by the definition of ∇' one has $\operatorname{im}\nabla'_q \subset \mathrm{H}^0(M,\nabla)^\perp = (\operatorname{im}\tilde{D})^{\perp\perp} = \operatorname{im}\tilde{D}$. Hence indeed $\operatorname{im}\tilde{D} = \operatorname{im}\nabla'_q$ as claimed. □

So far we have assumed that the connection potentials are smooth. But for the Fredholm property of the operators D and $\tilde{D}$ it actually suffices to assume a certain Sobolev regularity as in theorem 4.4. This follows from the fact that these operators are compact perturbations of the operators with smooth connection potentials. As previously one can determine the kernel and image of D and $\tilde{D}$ explicitly, which can then be used to establish theorem 4.3 and 4.4. For the first one only considers the operator $\tilde{D}$ on $W^{1,r}(M,E)$, establishes its Fredholm property, kernel and image, and exploits these to solve the equation. The proof of theorem 4.4 is similar but more complicated since one has to solve an equation for $\tilde{D}$ on $W^{1,q}(M,E)$ and D on $W^{2,p}(M,E)$ at the same time.

Proof of theorem 4.3 :
Let $2 \le r < \infty$ with $r > n$ and choose a finite bundle atlas of the vector bundle $V \hookrightarrow E$ over the n-manifold $M = \bigcup_{\alpha=1}^N U_\alpha$. Then the covariant derivative ∇ is given by connection potentials $A_\alpha \in W^{1,r}(U_\alpha, \mathrm{T}^*U_\alpha \otimes \operatorname{End} V)$. Now consider

$$\tilde{D}: \begin{array}{ccc} W^{1,r}(M,E) & \longrightarrow & W^{-1,r}(M,E) \\ u & \longmapsto & \nabla'\nabla u. \end{array}$$

We claim that this is well defined and a Fredholm operator of index 0. For that purpose let $\tilde{D}_0$ be the operator obtained by replacing ∇ in the above definition with a covariant derivative ∇_0 that has smooth connection potentials $A^0_\alpha \in \mathcal{C}^\infty(U_\alpha, \mathrm{T}^*U_\alpha \otimes \operatorname{End} V)$. Theorem 4.7 asserts that this is a Fredholm operator of index 0. so for the Fredholm property and index of $\tilde{D}$ it suffices to prove that $\tilde{D} - \tilde{D}_0$ is compact (see e.g. [Z, 5.8,Thm.5.E]). So we have to consider the separate terms in

$$\tilde{D} - \tilde{D}_0 = \nabla'\circ(\nabla - \nabla_0) + (\nabla' - \nabla'_0)\circ\nabla_0. \tag{4.11}$$

For the compactness of both terms one can use the fact that the composition of bounded and compact operators is again compact, so we establish the following:

(i) $\nabla - \nabla_0 : W^{1,r}(M,E) \to L^r(M, \mathrm{T}^*M \otimes E)$ is a compact operator.

This again is a composition of the compact embedding $W^{1,r} \hookrightarrow L^\infty$ with the bounded operator $\nabla - \nabla_0$ on $L^\infty(M,E)$. For the continuity of that operator it suffices to consider all $u \in \Gamma(E)$:

$$\|(\nabla - \nabla_0)u\|_r \;\leq\; C' \sum_\alpha \|A_\alpha - A^0_\alpha\|_r \|u_\alpha\|_\infty \;\leq\; C\|u\|_\infty .$$

Here the constants C' and C are finite due to remark B.1 and since the A_α are of class L^r.

(ii) $\nabla' - \nabla'_0 : L^r(M, \mathrm{T}^*M \otimes E) \to W^{-1,r}(M,E)$ is a compact operator.

A theorem of Schauder (e.g. [Z, 5.1,Thm.5.A]) says that the dual operator of a compact operator between Banach spaces is again compact. This is the dual operator of $\nabla - \nabla_0 : W^{1,r^*}(M,E) \to L^{r^*}(M, \mathrm{T}^*M \otimes E)$, so we only have to establish its compactness.

Let $\frac{1}{s} = \frac{1}{r^*} - \frac{1}{r} = 1 - \frac{2}{r}$, then the embedding $W^{1,r^*} \hookrightarrow L^s$ is compact since due to $r > n$

$$\tfrac{1}{n} - \tfrac{1}{r^*} \;=\; \tfrac{1}{n} - 1 + \tfrac{1}{r} \;>\; -1 + \tfrac{2}{r} \;=\; -\tfrac{1}{s}.$$

The operator $\nabla - \nabla_0$ is bounded on $L^s(M,E)$ since for all $u \in \Gamma(E)$ and some finite constants C, C'

$$\|(\nabla - \nabla_0)u\|_{r^*} \;\leq\; C' \sum_\alpha \|A_\alpha - A^0_\alpha\|_r \|u_\alpha\|_s \;\leq\; C\|u\|_s .$$

Firstly, these estimates show that $\tilde{D}$ indeed maps into $W^{-1,r}(M,E)$. Moreover, $\nabla' : L^r(M, \mathrm{T}^*M \otimes E) \to W^{-1,r}(M,E)$ is bounded by (ii). Hence both terms in the decomposition (4.11) of $\tilde{D} - \tilde{D}_0$ are compositions of a bounded and a compact map, and thus $\tilde{D} - \tilde{D}_0$ is in fact compact. This proves that $\tilde{D}$ is a Fredholm operator of index 0.

Now consider $u \in W^{1,r}(M,E)$ such that $\tilde{D}u = 0$. Note that $r^* \leq r$ due to $r \geq 2$, so one can test this equation with $u \in W^{1,r^*}(M,E)$ to obtain

$$0 \;=\; (\tilde{D}u)(u) \;=\; (\nabla'\nabla u)(u) \;=\; \int_M \langle\, \nabla u \wedge *\nabla u \,\rangle \,.$$

This shows that $\nabla u = 0$ and hence the kernel of $\tilde{D}$ is the set of horizontal sections,

$$\ker \tilde{D} \;=\; \mathrm{H}^0(M, \nabla) \;:=\; \left\{ u \in W^{1,r}(M,E) \;\middle|\; \nabla u = 0 \right\}.$$

Note that horizontal sections are not necessarily smooth in this case. Next, one can use the index of $\tilde{D}$ to determine its image:

$$\operatorname{im} \tilde{D} \;=\; \left\{ \nabla'\tau \;\middle|\; \tau \in L^r(M, \mathrm{T}^*M \otimes E) \right\} \;=:\; Z.$$

One has $\operatorname{im} \tilde{D} \subset Z$ since ∇ maps $W^{1,r}(M,E)$ to $L^r(M, \mathrm{T}^*M \otimes E)$. Moreover, we know that the codimension of $\operatorname{im} \tilde{D}$ is equal to the (finite) dimension of $\ker \tilde{D} = \mathrm{H}^0(M,\nabla)$, so for the equality of the two spaces it remains to prove that the codimension of Z is greater or equal to the dimension of $H^0(M,\nabla)$. This then follows from

$$\mathrm{H}^0(M,\nabla) \;\subset\; Z^\perp \;=\; \left\{\nabla'\tau \,\middle|\, \tau\in L^r(M,\mathrm{T}^*M\otimes E)\right\}^\perp,$$

which holds since for all $u\in\mathrm{H}^0(M,\nabla)$

$$(\nabla'\tau)(u) \;=\; \int_M \langle\, \tau\wedge *\nabla u\,\rangle \;=\; 0.$$

Now $\ker\tilde{D} = \mathrm{H}^0(M,\nabla)$ is finite dimensional, hence there exists a topological complement (e.g. by [Z, 3.9,Ex.17])

$$W^{1,r}(M,E) = \mathrm{H}^0(M,\nabla)\oplus Y.$$

Moreover, $\operatorname{im}\tilde{D}$ is closed, so the restriction $\tilde{D}: Y\to\operatorname{im}\tilde{D}$ is a bounded isomorphism between Banach spaces. Then by lemma E.3 (ii) it has a bounded inverse, i.e. there exists a constant C such that

$$\|u\|_{W^{1,r}} \le C\|\tilde{D}u\|_{W^{-1,r}} \qquad \forall u\in Y$$

Now let any $\tau\in W^{1,r}(M,\mathrm{T}^*M\otimes E)$ be given, then $\nabla'\tau\in\operatorname{im}\tilde{D}$, so there exists a preimage $u\in Y\subset W^{1,r}(M,E)$. This means that $\nabla'\nabla u=\nabla'\tau$, and by the previous estimate

$$\|u\|_{W^{1,r}} \;\le\; C\|\tilde{D}u\|_{W^{-1,r}} \;=\; C\|\nabla'\tau\|_{W^{-1,r}}.$$

□

Proof of theorem 4.4 and remark 4.5 :
Let $1<p\le q<\infty$ be as assumed for the theorem, that is

$$p>\tfrac{n}{2} \qquad\text{and}\qquad \tfrac{1}{n}>\tfrac{1}{q}\ge\tfrac{1}{p}-\tfrac{1}{n}.$$

Choose a finite bundle atlas of the vector bundle $V\hookrightarrow E$ over $M=\bigcup_{\alpha=1}^N U_\alpha$, then the covariant derivative under consideration, ∇, is given by connection potentials $A_\alpha\in W^{1,p}(U_\alpha,\mathrm{T}^*U_\alpha\otimes\operatorname{End}V)$. We will first show that the following are still well defined Fredholm operators of index 0 :

$$D:\begin{array}{ccc} W^{2,p}(M,E) & \longrightarrow & L^p(M,E)\times W^{1,p}_\partial(M,E)\\ u & \mapsto & (\nabla^*\nabla u\,,\,\nabla_\nu u),\end{array}$$

$$\tilde{D}:\begin{array}{ccc} W^{1,q}(M,E) & \longrightarrow & W^{-1,q}(M,E)\\ u & \mapsto & \nabla'\nabla u.\end{array}$$

Let ∇_0 be a covariant derivative on E that has smooth connection potentials $A^0_\alpha\in\mathcal{C}^\infty(U_\alpha,\mathrm{T}^*U_\alpha\otimes\operatorname{End}V)$ and denote by D_0 and $\tilde{D}_0$ the operators obtained by replacing ∇ with ∇_0 in above definition. These are Fredholm operators of index 0 by theorem 4.6 and 4.7.

We will see that D and $\tilde{D}$ are compact perturbations of D_0 and $\tilde{D}_0$, hence they themselves are Fredholm operators of index 0 (see e.g. [Z, 5.8,Thm.5.E]). In both cases the operators are composed of covariant derivatives, their adjoint and dual operators, so let us first consider differences of those operators:

(i) $\nabla - \nabla_0 : W^{2,p}(M,E) \to W^{1,p}(M, \mathrm{T}^*M \otimes E)$ is a compact operator.

Indeed, it is the composition of the compact embedding $W^{2,p} \hookrightarrow W^{1,2p}$ with the bounded operator $\nabla - \nabla_0$ on $W^{1,2p}(M,E)$. For the Sobolev embedding one checks that $1 - \frac{n}{p} > -\frac{1}{2p}$ due to $p > \frac{n}{2}$. For the continuity of the second operator it suffices to consider all $u \in \Gamma(E)$:

$$\begin{aligned}\|(\nabla - \nabla_0)u\|_{W^{1,p}} &\leq C' \sum_\alpha \|(A_\alpha - A_\alpha^0)u_\alpha\|_{W^{1,p}} \\ &\leq \sum_\alpha C_\alpha \|A_\alpha - A_\alpha^0\|_{W^{1,p}} \|u_\alpha\|_{W^{1,2p}} \\ &\leq C\|u\|_{W^{1,2p}}.\end{aligned}$$

In the first step, C' is finite due to remark B.1. In the second step one obtains finite constants C_α from lemma B.3 with $r = p$ and $s = 2p$ (it applies due to $2p > n$). This leads to a finite constant C since the A_α are of class $W^{1,p}$.

(ii) $\nabla^* - \nabla_0^* : W^{1,p}(M, \mathrm{T}^*M \otimes E) \to L^p(M,E)$ is a compact operator.

This is composed of the compact embedding $W^{1,p} \hookrightarrow L^{2p}$ and a bounded operator: There exist constants C, C' such that for all $\tau \in \Gamma(\mathrm{T}^*M \otimes E)$

$$\begin{aligned}\|(\nabla^* - \nabla_0^*)\tau\|_p &\leq C' \sum_\alpha \big\| {*}\big((A_\alpha - A_\alpha^0)^* \wedge {*}\tau_\alpha\big)\big\|_p \\ &\leq C' \sum_\alpha \|A_\alpha - A_\alpha^0\|_{2p} \|\tau_\alpha\|_{2p} \\ &\leq C\|\tau\|_{2p}.\end{aligned}$$

(iii) $\nabla - \nabla_0 : \ W^{1,q}(M,E) \to \ L^q(M, \mathrm{T}^*M \otimes E)$ and
$\nabla - \nabla_0 : W^{1,q^*}(M,E) \to L^{q^*}(M, \mathrm{T}^*M \otimes E)$ are both compact operators.

This is however not true on every space $W^{1,r}(M,E)$. In the first case one has the compact embedding $W^{1,q} \hookrightarrow \mathcal{C}^0$ due to $q > n$ and there exist constants C, C' such that for all $u \in \Gamma(E)$

$$\begin{aligned}\|(\nabla - \nabla_0)u\|_q &\leq C' \sum_\alpha \|(A_\alpha - A_\alpha^0)u_\alpha\|_q \\ &\leq C' \sum_\alpha \|A_\alpha - A_\alpha^0\|_q \|u_\alpha\|_\infty \\ &\leq C\|u\|_\infty.\end{aligned}$$

Here one also needs $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{n}$ in order that the Sobolev inequality for $W^{1,p} \hookrightarrow L^q$ holds and hence the $\|A_\alpha - A^0_\alpha\|_q$ are finite.

In the second case one has a compact embedding $W^{1,q^*} \hookrightarrow L^s$ and a continuous embedding $W^{1,p} \hookrightarrow L^{p'}$ with $\frac{1}{q^*} = \frac{1}{p'} + \frac{1}{s}$. In case $n = 1$ this is achieved by $p' = s = 2q^*$ since the Sobolev embedding $W^{1,r} \hookrightarrow \mathcal{C}^0$ is compact for all $r > 1$. In case $n \geq 2$ one chooses $p' = 2p$ and $\frac{1}{s} = \frac{1}{q^*} - \frac{1}{2p}$. This is possible since due to $q > n$ and $p > \frac{n}{2}$

$$\tfrac{1}{q^*} - \tfrac{1}{2p} \;=\; 1 - \tfrac{1}{q} - \tfrac{1}{2p} \;>\; 1 - \tfrac{1}{n} - \tfrac{1}{n} \;\geq\; 0.$$

Now $W^{1,p} \hookrightarrow L^{2p}$ holds as before and for the first embedding one uses $p > \frac{n}{2}$ to check

$$\tfrac{1}{n} - \tfrac{1}{q^*} \;>\; \tfrac{1}{2p} - \tfrac{1}{q^*} \;=\; -\tfrac{1}{s}.$$

Then use the Sobolev inequality for $W^{1,p} \hookrightarrow L^{p'}$ to obtain for all $u \in \Gamma(E)$ and some finite constants C, C'

$$\|(\nabla - \nabla_0)u\|_{q^*} \;\leq\; C' \sum_\alpha \|A_\alpha - A^0_\alpha\|_{p'} \|u_\alpha\|_s \;\leq\; C\|u\|_s.$$

(iv) $\nabla' - \nabla'_0 : L^q(M, \mathrm{T}^*M \otimes E) \to W^{-1,q}(M, E)$ is a compact operator.

This is the dual operator of $\nabla - \nabla_0 : W^{1,q^*}(M,E) \to L^{q^*}(M, \mathrm{T}^*M \otimes E)$, which is compact by (iii). Now by a theorem of Schauder (see e.g. [Z, 5.1,Thm.5.A]) the dual operator of a compact operator between Banach spaces is again compact.

Firstly, one can see from above estimates that D and $\tilde{D}$ are well defined, i.e. they indeed map into the spaces that are indicated. For the second component of D we should mention in addition that the map

$$\begin{array}{ccc} W^{1,p}(M, \mathrm{T}^*M \otimes E) & \longrightarrow & W^{1,p}_\partial(M,E) \\ \tau & \mapsto & \tau(\nu) \end{array}$$

is continuous and independent of the choice of the covariant derivative. This is due to the fact that ν can be extended to a smooth vector field $\tilde{\nu}$ on all of M and then $\tau(\nu) = \tau(\tilde{\nu})|_{\partial M}$.

Secondly, we can now check that $D - D_0$ and $\tilde{D} - \tilde{D}_0$ are compact since they are given by compositions of bounded and compact operators. The first component of $D - D_0$ is $\nabla^* \circ (\nabla - \nabla_0) + (\nabla^* - \nabla^*_0) \circ \nabla_0$, where $\nabla^* : W^{1,p}(M, \mathrm{T}^*M \otimes E) \to L^p(M,E)$ is bounded by (ii). The second component of $D - D_0$ is a composition of $\nabla - \nabla_0$ with above continuous map to $W^{1,p}_\partial(M,E)$.

Analogously, we can write $\tilde{D} - \tilde{D}_0 = \nabla' \circ (\nabla - \nabla_0) + (\nabla' - \nabla'_0) \circ \nabla_0$, where $\nabla' : L^q(M, \mathrm{T}^*M \otimes E) \to W^{-1,q}(M,E)$ is bounded since it is the dual of a bounded operator (see (iii)). This establishes the Fredholm property and index 0 for both D and $\tilde{D}$.

As next step we will show that the kernel of both operators again is the set of horizontal sections,

$$\ker D \;=\; \ker \tilde{D} \;=\; \mathrm{H}^0(M,\nabla) \;:=\; \bigl\{ u \in W^{2,p}(M,E) \,\big|\, \nabla u = 0 \bigr\}.$$

Note that horizontal sections are not necessarily smooth in this case. However, if $u \in W^{1,p}(M,E)$ satisfies $\nabla u = 0$ then $\mathrm{d}u_\alpha = -A_\alpha u_\alpha \in W^{1,p}(U_\alpha, \mathrm{T}^*U_\alpha \otimes V)$ in all bundle charts and hence in fact $u \in W^{2,p}(M,E)$. So for $u \in \ker D$ it remains to obtain $\nabla u = 0$ from lemma 4.1 :

$$\int_M \langle\, \nabla u \wedge *\nabla u \,\rangle \;=\; \int_M \langle\, u \,,\, \nabla^*\nabla u \,\rangle + \int_{\partial M} \langle\, u \,,\, \nabla_\nu u \,\rangle \;=\; 0.$$

This works via approximation by smooth sections due to the Sobolev embeddings $W^{2,p} \hookrightarrow L^\infty$ and $W^{1,p} \hookrightarrow L^2$ and it proves that $\ker D = \mathrm{H}^0(M,\nabla)$.

Moreover, we know from the index of D that $(\mathrm{im}\, D)^\perp$ has the same dimension as $\mathrm{H}^0(M,\nabla)$. But

$$\bigl\{ (u, u|_{\partial M}) \,\big|\, u \in \mathrm{H}^0(M,\nabla) \bigr\} \subset \bigl(L^p(M,E) \times W^{1,p}_\partial(M,E) \bigr)^*$$

annihilates $\mathrm{im}\, D$ by lemma 4.1, so for dimensional reasons it is identical to $(\mathrm{im}\, D)^\perp$. As in the proof of theorem 4.7 this helps to determine $\ker \tilde{D}$: Note that the Sobolev embedding $W^{2,p} \hookrightarrow W^{1,q^*}$ holds due to $\frac{1}{q^*} = 1 - \frac{1}{q} > \frac{1}{p} - \frac{1}{n}$. So if $u \in \ker \tilde{D}$ then for all $v \in W^{2,p}(M,E)$ one obtains from lemma 4.1

$$0 \;=\; \nabla'\nabla u(v) \;=\; \int_M \langle\, \nabla u \wedge *\nabla v \,\rangle \;=\; \int_M \langle\, u \,,\, \nabla^*\nabla v \,\rangle + \int_{\partial M} \langle\, u \,,\, \nabla_\nu v \,\rangle\,.$$

This implies $(u, u|_{\partial M}) \in (\mathrm{im}\, D)^\perp$ and thus $\ker \tilde{D} \subset \mathrm{H}^0(M,\nabla)$. Conversely, $\mathrm{H}^0(M,\nabla) \subset \ker \tilde{D}$ by the Sobolev embedding $W^{2,p} \hookrightarrow W^{1,q}$, so we have also proven $\ker \tilde{D} = \mathrm{H}^0(M,\nabla)$.

Furthermore, we determine the images of the operators using their index:

$$\begin{aligned} \mathrm{im}\, D &= \bigl\{ (\nabla^*\tau, \tau(\nu)) \,\big|\, \tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E) \bigr\}, \\ \mathrm{im}\, \tilde{D} &= \bigl\{ \nabla'\tau \,\big|\, \tau \in L^q(M, \mathrm{T}^*M \otimes E) \bigr\}. \end{aligned}$$

In both cases it is clear that the image is a subset of the other given space. For D this is since ∇ maps $W^{2,p}(M,E)$ to $W^{1,p}(M, \mathrm{T}^*M \otimes E)$ – see (i), and for $\tilde{D}$ we know from (iii) that ∇ maps $W^{1,q}(M,E)$ to $L^q(M, \mathrm{T}^*M \otimes E)$. So it remains to prove that the codimension of the given space is greater or equal to the codimension of the image, that equals $\dim \mathrm{H}^0(M,\nabla)$. One proves this by checking that $H^0(M,\nabla)$ lies in the annihilator of the given space:

For all $\tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E)$ and $v \in \mathrm{H}^0(M,\nabla)$ use lemma 4.1 to obtain

$$\int_M \langle\, v \,,\, \nabla^*\tau \,\rangle + \int_{\partial M} \langle\, v \,,\, \tau(\nu) \,\rangle \;=\; \int_M \langle\, \nabla v \wedge *\tau \,\rangle \;=\; 0.$$

For all $\tau \in W^{1,q}(M, \mathrm{T}^*M \otimes E)$ and $v \in \mathrm{H}^0(M, \nabla)$ one has

$$(\nabla'\tau)(u) \;=\; \int_M \langle\, \tau \wedge *\nabla u \,\rangle \;=\; 0.$$

Thus the images of the operators are established as claimed above.

With the help of the operators D and $\tilde{D}$ we can now give the construction that for all $\tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E)$ provides the claimed solution $u \in W^{2,p}(M, E)$ of

$$\begin{cases} \nabla^*\nabla u = \nabla^*\tau, \\ \;\;\nabla_\nu u = \tau(\nu) \end{cases} \quad \text{with} \quad \begin{array}{l} \|u\|_{W^{2,p}} \le C\big(\|\nabla^*\tau\|_p + \|\tau(\nu)\|_{W^{1,p}_\partial}\big), \\ \|u\|_{W^{1,q}} \le C\|\tau\|_q. \end{array}$$

Choose a topological complement Y of $\mathrm{H}^0(M, \nabla)$ in $W^{1,q}(M, E)$ (which is possible e.g. by [Z, 3.9,Ex.17] since we consider the kernel of a Fredholm operator, i.e. a finite dimensional subspace of a Banach space),

$$W^{1,q}(M, E) = \mathrm{H}^0(M, \nabla) \oplus Y. \tag{4.12}$$

Since $\mathrm{H}^0(M, \nabla)$ is in fact a subspace of $W^{2,p}(M, E)$ this also provides a topological complement of $\mathrm{H}^0(M, \nabla)$ in $W^{2,p}(M, E)$:

$$W^{2,p}(M, E) = \mathrm{H}^0(M, \nabla) \oplus \big(Y \cap W^{2,p}(M, E)\big). \tag{4.13}$$

The restrictions $\tilde{D} : Y \xrightarrow{\sim} \operatorname{im} \tilde{D}$ and $D : Y \cap W^{2,p}(M, E) \xrightarrow{\sim} \operatorname{im} D$ are (bounded) isomorphisms between Banach spaces. So by lemma E.3 (ii) their inverses also are bounded linear operators, hence there exists a constant C such that

$$\begin{array}{ll} \|u\|_{W^{1,q}} \le C\|\tilde{D}u\|_{W^{-1,q}} & \forall u \in Y \\ \|u\|_{W^{2,p}} \le C\|Du\|_{L^p \times W^{1,p}_\partial} & \forall u \in Y \cap W^{2,p}(M, E). \end{array} \tag{4.14}$$

Now let any $\tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E)$ be given, then $\big(\nabla^*\tau, \tau(\nu)\big) \in \operatorname{im} D$, thus there exists a preimage $u \in Y \cap W^{2,p}(M, E)$, i.e. $\nabla^*\nabla u = \nabla^*\tau$ and $\nabla_\nu u = \tau(\nu)$. The first estimate then follows directly from (4.14),

$$\|u\|_{W^{2,p}} \;\le\; C\|Du\|_{L^p \times W^{1,p}_\partial} \;=\; C\big(\|\nabla^*\tau\|_p + \|\tau(\nu)\|_{W^{1,p}_\partial}\big).$$

For the second estimate note that for all $v \in W^{1,q^*}(M, E)$ by lemma 4.1

$$\begin{aligned} (\tilde{D}u)(v) &= \int_M \langle\, \nabla u \wedge *\nabla v \,\rangle \\ &= \int_M \langle\, \nabla^*\nabla u \,,\, v \,\rangle + \int_{\partial M} \langle\, \nabla_\nu u \,,\, v \,\rangle \\ &= \int_M \langle\, \nabla^*\tau \,,\, v \,\rangle + \int_{\partial M} \langle\, \tau(\nu) \,,\, v \,\rangle \\ &= \int_M \langle\, \tau \wedge *\nabla v \,\rangle\,. \end{aligned}$$

Hence we have $|(\tilde{D}u)(v)| \le \|\tau\|_q \|\nabla v\|_{q^*}$, where $\|\nabla v\|_{q^*} \le \|v\|_{W^{1,q^*}}$, and thus from (4.14)

$$\begin{aligned}
\|u\|_{W^{1,q}} &\le C\|\tilde{D}u\|_{W^{-1,q}} \\
&= C \cdot \sup\left\{ \frac{|(\tilde{D}u)(v)|}{\|v\|_{W^{1,q^*}}} \,\middle|\, v \in W^{1,q^*}(M,E) \right\} \\
&\le C\|\tau\|_q.
\end{aligned}$$

This proves the theorem.

In remark 4.5 one considers different metrics g and g' on M. These lead to different operators D, $\tilde{D}$, and D', $\tilde{D}'$ with the same Fredholm properties as established above and the same kernel $\mathrm{H}^0(M,\nabla)$ – which is independent of the metric. Thus one can use the same topological complement Y in (4.12) and (4.13) for all metrics. So to establish remark 4.5 it remains to see that all constants in the above estimates can be chosen independent of the metric g' as long as $\|g-g'\|_{W^{1,\infty}} \le \varepsilon$, where $\varepsilon > 0$ can be chosen appropriately.

Firstly, we have to consider the constants in (4.14): Let C_g be the constant for the metric g, then we claim that for sufficiently small $\varepsilon > 0$ in $\|g-g'\|_{W^{1,\infty}} \le \varepsilon$ and with the constant C replaced by $2C_g$ the estimates (4.14) also hold for D' and $\tilde{D}'$. To see this for $\tilde{D}'$ first observe that for all $u \in Y$ and $v \in W^{1,q^*}(M,E)$

$$\begin{aligned}
\left|(\tilde{D}u - \tilde{D}'u)(v)\right| &= \left| \int_M \langle\, \nabla u \wedge *_g \nabla v \,\rangle - \int_M \langle\, \nabla u \wedge *_{g'} \nabla v \,\rangle \right| \\
&\le \|g^{-1} - g'^{-1}\|_\infty \|u\|_{W^{1,q}} \|v\|_{W^{1,q^*}}.
\end{aligned}$$

Here g^{-1} denotes the inner product on 1-forms induced by the metric g; in local coordinates it is given by the inverse matrix of g. One can choose $\varepsilon \le \frac{1}{2\|g^{-1}\|_\infty}$ such that (E.4) gives for all metrics g' with $\|g-g'\|_\infty \le \varepsilon$

$$\|g'^{-1}\|_\infty \le 2\|g^{-1}\|_\infty, \qquad \|g^{-1} - g'^{-1}\|_\infty \le C\|g-g'\|_\infty. \tag{4.15}$$

Here the constant is $C = 2\|g^{-1}\|_\infty^2$. Now (4.14) for the metric g leads to

$$\begin{aligned}
\|u\|_{W^{1,q}} &\le C_g\|\tilde{D}'u\|_{W^{-1,q}} + C_g\|\tilde{D}u - \tilde{D}'u\|_{W^{-1,q}} \\
&\le C_g\|\tilde{D}'u\|_{W^{-1,q}} + C_g \sup\left\{ \frac{|(\tilde{D}u - \tilde{D}'u)(v)|}{\|v\|_{W^{1,q^*}}} \,\middle|\, v \in W^{1,q^*}(M,E) \right\} \\
&\le C_g\|\tilde{D}'u\|_{W^{-1,q}} + CC_g\|g-g'\|_\infty \|u\|_{W^{1,q}}.
\end{aligned}$$

If moreover $\varepsilon \le \frac{1}{2CC_g}$ in $\|g-g'\|_{W^{1,\infty}} \le \varepsilon$ then the last term can be taken to the left hand side and hence for all $u \in Y$

$$\|u\|_{W^{1,q}} \le 2C_g\|\tilde{D}'u\|_{W^{-1,q}}.$$

To obtain a uniform constant for the second estimate in (4.14) use this estimate for the metric g to obtain for all $u \in Y \cap W^{2,p}(M,E)$

$$\|u\|_{W^{2,p}} \leq C_g \|D'u\|_{L^p \times W^{1,p}_\partial} + C_g\big(\|(\nabla^*_g - \nabla^*_{g'})\nabla u\|_p + \|\nabla u(\nu_g - \nu_{g'})\|_{W^{1,p}_\partial}\big).$$

Here the last term is bounded by $\frac{1}{2}\|u\|_{W^{2,p}}$ for sufficiently small $\|g-g'\|_{W^{1,\infty}}$, so with a suitable choice of $\varepsilon > 0$ one obtains for all $u \in Y \cap W^{2,p}(M,E)$

$$\|u\|_{W^{2,p}} \leq 2C_g \|D'u\|_{L^p \times W^{1,p}_\partial}.$$

Indeed, $\|(\nabla^*_g - \nabla^*_{g'})\nabla u\|_p$ is – for a suitable choice of $\varepsilon > 0$ – bounded by an arbitrary small multiple of $\|u\|_{W^{2,p}}$. This is seen from the coordinate expression of ∇^*: It only contains the metric, its inverse, and first derivatives.

To estimate the $W^{1,p}_\partial$-norm of $\nabla u(\nu_g - \nu_{g'})$ extend ν_g to a smooth vector field $\tilde\nu_g$ on M. Note that in local coordinates $\nu^i_{g'} = g'^{ij} g_{jk} \nu^k_g$ and use this expression to also extend $\nu_{g'}$ to a vector field $\tilde\nu_{g'}$ on M. This construction shows that there is a constant C depending only on g such that

$$\|\nabla u(\nu_g - \nu_{g'})\|_{W^{1,p}_\partial} \;\leq\; \|\nabla u(\tilde\nu_g - \tilde\nu_{g'})\|_{W^{1,p}} \;\leq\; C\|g^{-1} - g'^{-1}\|_{W^{1,\infty}} \|u\|_{W^{2,p}}.$$

Now we have to make sure that the factor $\|g^{-1} - g'^{-1}\|_{W^{1,\infty}}$ in front of $\|u\|_{W^{2,p}}$ can be made small by the choice of $\varepsilon > 0$. Firstly, as before choose $\varepsilon \leq \frac{1}{2\|g^{-1}\|_\infty}$ such that (4.15) holds. Then moreover we find a constant C such that

$$\begin{aligned}
\|\nabla(g^{-1} - g'^{-1})\|_\infty &= \|g^{-1}\nabla g\, g^{-1} - g'^{-1}\nabla g'\, g'^{-1}\|_\infty \\
&\leq \|g^{-1} - g'^{-1}\|_\infty \|\nabla g\, g^{-1}\|_\infty + \|g'^{-1}\|_\infty \|\nabla g - \nabla g'\|_\infty \|g^{-1}\|_\infty \\
&\qquad + \|g'^{-1}\|_\infty \big(\|\nabla g\|_\infty + \|\nabla g - \nabla g'\|_\infty\big) \|g^{-1} - g'^{-1}\|_\infty \\
&\leq C\|g - g'\|_{W^{1,\infty}}.
\end{aligned}$$

Thus indeed, $\|g^{-1} - g'^{-1}\|_{W^{1,\infty}}$ can be made as small as we wish by the choice of ε in $\|g - g'\|_{W^{1,\infty}} \leq \varepsilon$. So we have established uniform constants in (4.14) for all metrics g' within some $W^{1,\infty}$-neighbourhood of g. For the first estimate of the theorem no further estimates are involved, so one obtains a uniform constant as claimed in remark 4.5. For the second estimate let $\tau \in W^{1,p}(M, \mathrm{T}^*M \otimes E)$ be given and let $u \in Y \cap W^{2,p}(M,E)$ be the solution of $D'u = (\nabla^*_{g'}\tau, \tau(\nu_{g'}))$. Then for all $v \in W^{1,q^*}(M,E)$

$$\big|(\tilde D' u)(v)\big| \;=\; \left|\int_M \langle\, \tau \wedge *_{g'} \nabla v \,\rangle\right| \;\leq\; \|g'^{-1}\|_\infty \|\tau\|_q \|v\|_{W^{1,q^*}},$$

and hence with (4.15) we find a constant C independent of g' such that

$$\|u\|_{W^{1,q}} \;\leq\; 2C_g \|\tilde D' u\|_{W^{-1,q}} \;\leq\; 2C_g \|g'^{-1}\|_\infty \|\tau\|_q \;\leq\; C\|\tau\|_q.$$

□

Part II

Weak Compactness

Chapter 5

Regularity for 1-Forms

In this chapter we establish a regularity result for 1-forms that generalizes the Hodge theory to manifolds with boundary. It moreover allows to obtain regularity and estimates for separate components of the 1-forms. The reason for the discussion of this componentwise Hodge type regularity result is that it has two important corollaries which will be needed for the proof of Uhlenbeck's compactness theorems. Firstly, the proof of the weak Uhlenbeck compactness theorem centrally relies on the L^p-estimate in theorem D for the operator $\mathrm{d} \oplus \mathrm{d}^*$ on 1-forms. In [U2] only the L^2-estimate for the Euclidean unit ball is proven explicitly. In this chapter we will obtain the subsequent general L^p-result. Note that the case $*A|_{\partial M} = 0$ is theorem D.

Here and throughout this chapter M is a compact, oriented Riemannian manifold with (possibly empty) boundary ∂M.

Theorem 5.1 *For every $1 < p < \infty$ there exists a constant C such that the following holds.*

(i) Suppose that $A \in W^{1,p}(M, \mathrm{T}^*M)$ satisfies $*A|_{\partial M} = 0$ or $A|_{\partial M} = 0$. Then

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p + \|A\|_p\big).$$

(ii) Assume in addition $H^1(M;\mathbb{R}) = 0$ and suppose that $A \in W^{1,p}(M, \mathrm{T}^*M)$ satisfies $*A|_{\partial M} = 0$, then

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p\big).$$

Moreover, the constants can be chosen such that they depend continuously on the metric with respect to the $W^{1,\infty}$-topology.

Secondly, the regularity theory in chapter 9 for the (weak) Yang-Mills equation in conjunction with the (relative) Coulomb gauge will be obtained from the subsequent theorem. Here $\langle \cdot, \cdot \rangle$ denotes the pointwise inner product between differential forms, $\langle \alpha, \beta \rangle = *(\alpha \wedge *\beta)$.

Theorem 5.2 *For every $k \in \mathbb{N}$ and $1 < p < \infty$ there exists a constant C such that the following holds: Suppose that a 1-form $\alpha \in W^{k,p}(M, \mathrm{T}^*M)$ satisfies for some $G \in W^{k,p}(M)$*

$$\begin{cases} \mathrm{d}^*\alpha = G, \\ *\alpha|_{\partial M} = 0, \end{cases} \tag{5.1}$$

and that for some $\gamma \in W^{k-1,p}(M, \mathrm{T}^*M)$ and an $\omega \in W^{k,p}(M, \Lambda^2\mathrm{T}^*M)$

$$\int_M \langle\, \mathrm{d}\alpha\,,\, \mathrm{d}\beta\,\rangle = \int_M \langle\, \gamma\,,\, \beta\,\rangle + \int_{\partial M} \beta \wedge *\omega \tag{5.2}$$

holds for all smooth $\beta \in \Omega^1(M)$. Then $\alpha \in W^{k+1,p}(M, \mathrm{T}^*M)$ and

$$\|\alpha\|_{W^{k+1,p}} \le C\big(\|\gamma\|_{W^{k-1,p}} + \|\omega\|_{W^{k,p}} + \|G\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\big).$$

Moreover, the constant C can be chosen such that it depends continuously on the metric with respect to the $W^{k+1,\infty}$-topology.

Both theorems 5.1 and 5.2 will be corollaries of the general regularity theorem 5.3 below. Before we state it, let us consider the case $*\alpha|_{\partial M} = 0$ in the last theorem. If $\alpha \in W^{2,p}(M, \mathrm{T}^*M)$ then (5.1) and (5.2) are equivalent to the following boundary value problem:

$$\begin{cases} \mathrm{d}^*\alpha \in W^{k,p}(M, \mathrm{T}^*M), \\ \mathrm{d}^*\mathrm{d}\alpha \in W^{k-1,p}(M, \mathrm{T}^*M), \\ \alpha(\nu) = 0, \\ \iota_\nu \mathrm{d}\alpha \in W^{k,p}_\partial(M, \mathrm{T}^*M). \end{cases}$$

On the half space $\mathbb{H}$ with Euclidean metric and $\nu = \frac{\partial}{\partial x^0}$ this implies

$$\begin{cases} \Delta\alpha_i \in W^{k-1,p}(\mathbb{H}) & \forall i \ge 0, \\ \alpha_0 = 0, & \\ \partial_0\alpha_i - \partial_i\alpha_0 \in W^{k,p}_\partial(\mathbb{H}) & \forall i \ge 1. \end{cases}$$

Actually, in both above theorems, we deal with a weak form of these equations with additional lower order terms, but these simplified equations allow to see how the Dirichlet and Neumann problem are combined here: The regularity of the normal component α_0 follows from the fact that it solves a Dirichlet problem with homogenous boundary condition. Now the second boundary condition becomes an inhomogenous Neumann boundary condition for the tangential components, $\partial_0\alpha_i \in W^{k,p}_\partial(\mathbb{H})$. The actual proof will show that the regularity of the tangential components is in fact independent of the normal component.

By remark B.1 the proof of theorem 5.1 and 5.2 reduces to the separate regularity and estimates for the components of the 1-form. This regularity will be provided by theorem 5.3 below. This result is stated separately since it permits to obtain regularity and estimates for specific components of a 1-form by testing the weak equations with specific test functions, e.g. (5.2) may hold only for specific 1-forms β. Such weak equations with a restriction on the test 1-forms β occur when the Yang-Mills functional is minimized under certain boundary conditions, and this leads to weaker boundary conditions than the usual $*F_A|_{\partial M} = 0$. However, one can still obtain regularity results, see for example [W].

In the next theorem we avoid using 1-forms in the distributional sense by splitting and restricting to three relevant cases. Actually, the statement in case (i) will be true for all $k \in \mathbb{Z}$ if we use the definition of $W^{k,p}$ for $k < 0$ as dual space of W^{-k,p^*}. The theorem moreover deals with the following spaces of test functions:

$$\begin{aligned}\mathcal{C}^\infty_\delta(M) &:= \bigl\{\phi \in \mathcal{C}^\infty(M) \bigm| \phi|_{\partial M} = 0\bigr\},\\ \mathcal{C}^\infty_\nu(M) &:= \bigl\{\phi \in \mathcal{C}^\infty(M) \bigm| \tfrac{\partial\phi}{\partial\nu}|_{\partial M} = 0\bigr\}.\end{aligned}$$

Theorem 5.3 *Let $1 < p < \infty$ and fix an integer $k \geq -1$. Let $X \in \Gamma(\mathrm{T}M)$ be a smooth vector field and let either $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$ or $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$. Then there exists a constant C such that the following holds (in the cases $k \geq 1$, $k = 0$, and $k = -1$):*

(i) Let $k \geq 1$ and suppose that the 1-form $\alpha \in W^{k,p}(M, \mathrm{T}^*M)$ satisfies the following weak equation for some $f_1, f_2 \in W^{k-1,p}(M)$ and $h \in W^{k,p}_\delta(M)$

$$\begin{aligned}\int_M \langle\, \alpha\,,\, \mathrm{d}(\mathcal{L}_X\phi)\,\rangle &= \int_M f_1 \cdot \phi && \forall \phi \in \mathcal{T},\\ \int_M \langle\, \alpha\,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi)\,\rangle &= \int_M f_2 \cdot \phi + \int_{\partial M} h \cdot \phi && \forall \phi \in \mathcal{T}.\end{aligned}$$

Then $\alpha(X) \in W^{k+1,p}(M)$ and

$$\|\alpha(X)\|_{W^{k+1,p}} \leq C\bigl(\|f_1\|_{W^{k-1,p}} + \|f_2\|_{W^{k-1,p}} + \|h\|_{W^{k,p}_\delta} + \|\alpha\|_{W^{k,p}}\bigr).$$

(ii) Suppose that the 1-form $\alpha \in L^p(M, \mathrm{T}^*M)$ satisfies the following estimates with some constants A and B,

$$\begin{aligned}\Bigl|\int_M \langle\, \alpha\,,\, \mathrm{d}\mathcal{L}_X\phi\,\rangle\Bigr| &\leq A\|\phi\|_{W^{1,p^*}(M)} && \forall \phi \in \mathcal{T},\\ \Bigl|\int_M \langle\, \alpha\,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi)\,\rangle\Bigr| &\leq B\|\phi\|_{W^{1,p^*}(M)} && \forall \phi \in \mathcal{T}.\end{aligned}$$

Then $\alpha(X) \in W^{1,p}(M)$ and

$$\|\alpha(X)\|_{W^{1,p}} \leq C\bigl(A + B + \|\alpha\|_{L^p}\bigr).$$

(iii) Suppose that the 1*-form* $\alpha \in L^p(M, \mathrm{T}^*M)$ *satisfies the following estimates with some constants* A *and* B*,*

$$\Big|\int_M \langle\, \alpha\,,\, \mathrm{d}\mathcal{L}_X\phi \,\rangle\Big| \le A\|\phi\|_{W^{2,p^*}(M)} \qquad \forall \phi \in \mathcal{T},$$

$$\Big|\int_M \langle\, \alpha\,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi) \,\rangle\Big| \le B\|\phi\|_{W^{2,p^*}(M)} \qquad \forall \phi \in \mathcal{T}.$$

Then

$$\|\alpha(X)\|_{L^p} \le C\big(A + B + \|\alpha\|_{(W^{1,p}(M,\mathrm{T}^*M))^*}\big).$$

Remark 5.4 *The constants C in theorem 5.3 can be chosen such that they depend continuously on the metric and the vector field* X *with respect to the* $W^{k+1,\infty}$*-topology in case (i) and (ii) and the* $W^{2,\infty}$*-topology in case (iii).*

Before proving all these theorems let us also state the generalization of theorem 5.2 to componentwise regularity and estimates, since this will be useful in chapter 9. Here and in the following, the outer unit normal vector field to ∂M is denoted by ν, as in the previous part.

Theorem 5.5 *Let* $k \in \mathbb{N}$ *and* $1 < p < \infty$*. Let* $X \in \Gamma(\mathrm{T}M)$ *be a smooth vector field that is either perpendicular to the boundary, i.e.* $X|_{\partial M} = \psi \cdot \nu$ *for some* $\psi \in \mathcal{C}^\infty(\partial M)$*, or is tangential, i.e.* $X|_{\partial M} \in \Gamma(\mathrm{T}\partial M)$*. In the first case let* $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$*, in the latter case let* $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$*. Moreover, let* $N \subset \partial M$ *be an open subset such that* X *vanishes in a neighbourhood of* $\partial M \setminus N \subset M$*. Then there exists a constant* C *such that the following holds:*

Suppose that a 1*-form* $\alpha \in W^{k,p}(M, \mathrm{T}^*M)$ *satisfies for some* $G \in W^{k,p}(M)$

$$\begin{cases} \mathrm{d}^*\alpha = G, \\ *\alpha|_{\partial M} = 0 \quad \text{on } N, \end{cases} \tag{5.3}$$

and that for some $\gamma \in W^{k-1,p}(M, \mathrm{T}^*M)$ *and an* $\omega \in W^{k,p}(M, \Lambda^2\mathrm{T}^*M)$

$$\int_M \langle\, \mathrm{d}\alpha\,,\, \mathrm{d}\beta \,\rangle = \int_M \langle\, \gamma\,,\, \beta \,\rangle + \int_{\partial M} \beta \wedge *\omega \tag{5.4}$$

holds for all 1*-forms* $\beta = \phi \cdot \iota_X g$ *with* $\phi \in \mathcal{T}$*. Then* $\alpha(X) \in W^{k+1,p}(M)$ *and*

$$\|\alpha(X)\|_{W^{k+1,p}} \le C\big(\|\gamma\|_{W^{k-1,p}} + \|\omega\|_{W^{k,p}} + \|G\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\big).$$

Moreover, the constant C *can be chosen such that it depends continuously on the metric and the vector field* X *with respect to the* $W^{k+1,\infty}$*-topology.*

Depending on the behaviour of X on the boundary one obtains a different equation on $\alpha(X)$: In the normal case this will be a weak Dirichlet equation, in the tangential case it is a weak Neumann equation with an inhomogenous boundary condition, i.e. a boundary term. Note that the boundary condition $*\alpha|_{\partial M} = 0$ is only required on those parts of the boundary where the vector field X does not vanish identically. So in order to establish regularity on a subset of M, one does not need to multiply the 1-form α by a cutoff function but can simply consider appropriate vector fields X. (One uses the cutoff function on the vector field rather than on the 1-form.)

As a preparation for the proofs of the above theorems we first prove that the boundary condition $*\alpha|_{\partial M} = 0$ is equivalent to $\alpha(\nu) = 0$, and we establish general identities for the dual operator of the interior multiplication ι_X and of the Lie derivative $\mathcal{L}_X$ on functions and 1-forms. For that purpose denote the Riemannian metric on M by $g \in \Gamma(\mathrm{T}^*M \otimes \mathrm{T}^*M)$ and remember that this provides a duality between vector fields and 1-forms: Every $X \in \Gamma(\mathrm{T}M)$ defines a 1-form $\iota_X g \in \Omega^1(M)$, where ι_X denotes the interior multiplication, i.e. $\iota_X g(Y) = g(X,Y)$ for all $Y \in \Gamma(\mathrm{T}M)$. On the other hand, every 1-form $\alpha \in \Omega^1(M)$ uniquely determines a vector field $Y_\alpha \in \Gamma(\mathrm{T}M)$ by $\iota_{Y_\alpha} g = \alpha$.

Lemma 5.6 *Let $X \in \Gamma(\mathrm{T}M)$, then the following holds.*

(i) Let $\alpha \in \Omega^k(M)$, where $k \geq 1$, then

$$*\alpha|_{\partial M} = 0 \quad \text{on } N \iff \iota_\nu \alpha = 0 \quad \text{on } N$$

and for $\alpha \in \Omega^1(M)$

$$*\alpha|_{\partial M} = \alpha(\nu)\,\mathrm{dvol}_{\partial M}.$$

(ii) For all $\alpha \in \Omega^{k+1}(M)$ and $\beta \in \Omega^k(M)$, where $k \geq 0$,

$$\langle\, \iota_X \alpha\, ,\, \beta\,\rangle = \langle\, \alpha\, ,\, \iota_X g \wedge \beta\,\rangle\,.$$

(iii) For all $f, \phi \in \Omega^0(M)$

$$\int_M \mathcal{L}_X f \cdot \phi = -\int_M f \cdot \mathcal{L}_X \phi \; - \int_M \mathrm{div} X \cdot f \cdot \phi \; + \int_{\partial M} g(X,\nu) \cdot f \cdot \phi.$$

(iv) For all $\alpha, \beta \in \Omega^1(M)$ let $Y_\alpha \in \Gamma(TM)$ be given by $\iota_{Y_\alpha} g = \alpha$, then

$$\begin{aligned}\int_M \langle\, \mathcal{L}_X \alpha\, ,\, \beta\,\rangle = &-\int_M \langle\, \alpha\, ,\, \mathcal{L}_X \beta\,\rangle \; - \int_M \mathrm{div} X \;\langle\, \alpha\, ,\, \beta\,\rangle \\ &+ \int_M \langle\, \iota_{Y_\alpha}(\mathcal{L}_X g)\, ,\, \beta\,\rangle \; + \int_{\partial M} g(X,\nu) \cdot \langle\, \alpha\, ,\, \beta\,\rangle\,.\end{aligned}$$

Proof: To see (i) choose coordinates near a point in $N \subset \partial M$ such that $\nu = \frac{\partial}{\partial x^0}$ and $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ are the tangential directions. Then $*\alpha|_{\partial M} = 0$ at that point exactly means that the components $\alpha_{0 i_1 \ldots i_k}$ vanish for all $i_1, \ldots i_k$ and that just says $\iota_\nu \alpha = 0$ at that point. If $\alpha \in \Omega^1(M)$ then in fact

$$*\alpha|_{\partial M} \;=\; \alpha_0\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \;=\; \alpha(\nu)\,\mathrm{dvol}_{\partial M}.$$

(ii) also is a pointwise identity, so it suffices to consider geodesic coordinates at a fixed point. Then one has to prove the identity for all coordinate vector fields $X = \frac{\partial}{\partial x^\ell}$ and differential forms $\alpha = \mathrm{d}x^{i_0} \wedge \ldots \wedge \mathrm{d}x^{i_k}$ and $\beta = \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_k}$ with $i_0 < \ldots < i_k$ and $j_1 < \ldots < j_k$. Note that in this situation $\iota_X g = \mathrm{d}x^\ell$. Now first consider the case $(i_0, \ldots, i_k) = \sigma(\ell, j_1, \ldots, j_k)$ for some permutation σ of $k+1$ elements. In that case

$$\langle\, \iota_X \alpha \,,\, \beta \,\rangle \;=\; \operatorname{sign} \sigma \;=\; \langle\, \alpha \,,\, \iota_X g \wedge \beta \,\rangle \,.$$

If $(i_0, \ldots, i_k)$ contains other elements than $(\ell, j_1, \ldots, j_k)$ then both expressions $\langle\, \iota_X \alpha \,,\, \beta \,\rangle$ and $\langle\, \alpha \,,\, \iota_X g \wedge \beta \,\rangle$ vanish and this proves the equality.

For (iii) and (iv) first only consider one function $F \in \mathcal{C}^\infty(M)$ and calculate in local geodesic coordinates

$$\begin{aligned} \mathcal{L}_X F \;&=\; \sum_k X^k \partial_k F \;=\; \sum_k \partial_k (X^k F) - F \cdot \sum_k \partial_k X^k \\ &=\; *\mathrm{d}(F * \iota_X g) - F \cdot \operatorname{div} X. \end{aligned}$$

Then use Stokes' theorem and the fact $*\iota_X g|_{\partial M} = g(X, \nu)\, \mathrm{dvol}_{\partial M}$ from (i) to obtain

$$\begin{aligned} \int_M *\mathrm{d}(F * \iota_X g)\, \mathrm{dvol}_M &= \int_M \mathrm{d}(F * \iota_X g) \\ &= \int_{\partial M} F * \iota_X g \;=\; \int_{\partial M} F \cdot g(X, \nu)\, \mathrm{dvol}_{\partial M}. \end{aligned}$$

Putting this together we have shown for all $F \in \mathcal{C}^\infty(M)$

$$\int_M \mathcal{L}_X F = -\int_M \operatorname{div} X \cdot F \;+ \int_{\partial M} g(X, \nu) \cdot F.$$

To prove (iii) just apply this identity to $F = f \cdot \phi$ and note that

$$\mathcal{L}_X F = \mathcal{L}_X f \cdot \phi + f \cdot \mathcal{L}_X \phi.$$

For (iv) let $F = \langle\, \alpha \,,\, \beta \,\rangle$, then calculate in local coordinates

$$\begin{aligned} \mathcal{L}_X F &= \sum_{i,j,k} X^k \partial_k \big(g^{ij} \alpha_i \beta_j \big) \\ &= \sum_{i,j,k} g^{ij} X^k \partial_k \alpha_i \beta_j + \sum_{i,j,k} g^{ij} X^k \alpha_i \partial_k \beta_j - \sum_{i,j,k,\ell,m} g^{\ell j} X^k \partial_k g_{\ell m} g^{im} \alpha_i \beta_j \\ &= \langle\, \mathcal{L}_X \alpha \,,\, \beta \,\rangle + \langle\, \alpha \,,\, \mathcal{L}_X \beta \,\rangle - \langle\, \iota_{Y_\alpha} (\mathcal{L}_X g) \,,\, \beta \,\rangle \,. \end{aligned}$$

Here we used the formulae $(\mathcal{L}_X g)_{\ell m} = \sum_k X^k \partial_k g_{\ell m}$ and $(Y_\alpha)^m = g^{im} \alpha_i$ for the components of the vector field Y_α that is dual to α. $\square$

Proof of theorem 5.3 and remark 5.4:
Let $\alpha^\nu \in \mathcal{C}^\infty(M, \mathrm{T}^*M)$ be an L^p-approximating sequence for α such that $\alpha^\nu \equiv 0$ in a neighbourhood of ∂M. Then one obtains for all $\phi \in \mathcal{T}$

$$\begin{aligned}
\int_M \alpha(X) \cdot \Delta\phi &= \lim_{\nu\to\infty} \int_M \mathrm{d}\iota_X \alpha^\nu \cdot \mathrm{d}\phi \\
&= \lim_{\nu\to\infty} \Big(\int_M \langle\, \mathcal{L}_X \alpha^\nu \,,\, \mathrm{d}\phi \,\rangle - \int_M \langle\, \iota_X \mathrm{d}\alpha^\nu \,,\, \mathrm{d}\phi \,\rangle \Big) \\
&= \lim_{\nu\to\infty} \Big(- \int_M \langle\, \alpha^\nu \,,\, \mathcal{L}_X \mathrm{d}\phi \,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{div}X \cdot \mathrm{d}\phi \,\rangle \\
&\qquad + \int_M \langle\, \iota_{Y_{\alpha^\nu}} \mathcal{L}_X g \,,\, \mathrm{d}\phi \,\rangle - \int_M \langle\, \mathrm{d}\alpha^\nu \,,\, \iota_X g \wedge \mathrm{d}\phi \,\rangle \Big) \\
&= \lim_{\nu\to\infty} \Big(- \int_M \langle\, \alpha^\nu \,,\, \mathrm{d}(\mathcal{L}_X \phi) \,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi) \,\rangle \\
&\qquad + \int_M \big\langle\, \mathrm{d}^*\big(\iota_{Y_{\alpha^\nu}} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha^\nu\big) \,,\, \phi \,\big\rangle \Big) \\
&= - \int_M \langle\, \alpha \,,\, \mathrm{d}(\mathcal{L}_X \phi) \,\rangle - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi) \,\rangle \\
&\quad + \int_M \big\langle\, \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha\big) \,,\, \phi \,\big\rangle \\
&= \int_M \big(-f_1 - f_2 + \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha\big)\big)\, \phi \;-\; \int_{\partial M} h \cdot \phi
\end{aligned}$$

Here we used Cartan's formula $\mathcal{L}_X \alpha = \mathrm{d}\iota_X \alpha + \iota_X \mathrm{d}\alpha$, and the vector field Y_α is given by $\iota_{Y_\alpha} g = \alpha$. All boundary terms vanish since the α^ν vanish at the boundary.

Now in case $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$ the last boundary term also vanishes and the above calculation shows that $\alpha(X)$ is a weak solution of the Dirichlet problem (D.2) for $f = -f_1 - f_2 + \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha\big) \in W^{k-1,p}(M)$. Thus by the regularity theorem D.2 we have $\alpha(X) \in W^{k+1,p}(M)$ with the estimate

$$\begin{aligned}
\|\alpha(X)\|_{W^{k+1,p}} &\leq C \big\| -f_1 - f_2 + \mathrm{d}^*\big(\iota_{Y_\alpha} \mathcal{L}_X g \,-\, \mathrm{div}X \cdot \alpha\big) \big\|_{W^{k-1,p}} \\
&\leq C \big(\|f_1\|_{W^{k-1,p}} + \|f_2\|_{W^{k-1,p}} + \|\alpha\|_{W^{k,p}} \big).
\end{aligned}$$

Here C denotes any finite constant. In the first estimate, the constant from theorem D.2 depends continuously on the metric in the $W^{k,\infty}$-topology. In the second inequality, we pick up derivatives of g and X up to order $k+1$, so in the end the constant depends continuously on the metric g and the vector field X in the $W^{k+1,\infty}$-topology.

In case $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$ the above calculation shows that $\alpha(X)$ is a weak solution of the inhomogenous Neumann problem (3.4) for $f \in W^{k-1,p}(M)$ as before and with the boundary condition $h \in W^{k,p}_\delta(M)$. So the regularity theorem 3.2 asserts that $\alpha(X) \in W^{k+1,p}(M)$ with the estimate

$$\begin{aligned}\|\alpha(X)\|_{W^{k+1,p}} &\le C\big(\big\|-f_1 - f_2 + \mathrm{d}^*\big(\iota_{Y_\alpha}\mathcal{L}_X g \,-\, \mathrm{div}X\cdot\alpha\big)\big\|_{W^{k-1,p}} + \|h\|_{W^{k,p}_\delta}\big)\\ &\le C\big(\|f_1\|_{W^{k-1,p}} + \|f_2\|_{W^{k-1,p}} + \|h\|_{W^{k,p}_\delta} + \|\alpha\|_{W^{k,p}}\big).\end{aligned}$$

Again, in the first estimate, the constant from theorem 3.2 depends continuously on the metric in the $W^{k,\infty}$-topology, but in the second inequality, the derivatives of g and X lead to continuous $W^{k+1,\infty}$-dependence of the constant on the metric and the vector field.

In case (ii), i.e. for $k = 0$, we do a similar calculation as above for all $\phi \in \mathcal{T}$:

$$\begin{aligned}\int_M \alpha(X)\cdot\Delta\phi &= \lim_{\nu\to\infty}\Big(\int_M \langle\, \mathcal{L}_X\alpha^\nu \,,\, \mathrm{d}\phi\,\rangle - \int_M \langle\, \iota_X \mathrm{d}\alpha^\nu \,,\, \mathrm{d}\phi\,\rangle\Big)\\ &= \lim_{\nu\to\infty}\Big(-\int_M \langle\, \alpha^\nu \,,\, \mathcal{L}_X\mathrm{d}\phi\,\rangle - \int_M \langle\, \alpha^\nu \,,\, \mathrm{div}X\cdot\mathrm{d}\phi\,\rangle\\ &\qquad - \int_M \big\langle\, \alpha^\nu \,,\, \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g\,\big\rangle - \int_M \langle\, \mathrm{d}\alpha^\nu \,,\, \iota_X g\wedge\mathrm{d}\phi\,\rangle\Big)\\ &= -\int_M \langle\, \alpha \,,\, \mathrm{d}(\mathcal{L}_X\phi)\,\rangle - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g\wedge\mathrm{d}\phi)\,\rangle\\ &\quad - \int_M \big\langle\, \alpha \,,\, \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g \,-\, \mathrm{div}X\cdot\mathrm{d}\phi\big)\,\big\rangle\\ &\le A\|\phi\|_{W^{1,p^*}} + B\|\phi\|_{W^{1,p^*}} + \|\alpha\|_{L^p}\|\iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g - \mathrm{div}X\cdot\mathrm{d}\phi\|_{L^{p^*}}\\ &\le \big(A + B + C\|\alpha\|_{L^p}\big)\|\phi\|_{W^{1,p^*}}\end{aligned}$$

Here the constant C varies continuously with the metric g and the vector field X in the $W^{1,\infty}$-topology. In case (iii), i.e. for $k = -1$, the same calculation gives for all $\phi \in \mathcal{T}$

$$\begin{aligned}&\int_M \alpha(X)\cdot\Delta\phi\\ &= -\int_M \langle\, \alpha \,,\, \mathrm{d}(\mathcal{L}_X\phi)\,\rangle - \int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g\wedge\mathrm{d}\phi)\,\rangle - \int_M \big\langle\, \alpha \,,\, \iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g \,-\, \mathrm{div}X\cdot\mathrm{d}\phi\big)\,\big\rangle\\ &\le A\|\phi\|_{W^{2,p^*}} + B\|\phi\|_{W^{2,p^*}} + \|\alpha\|_{(W^{1,p^*})^*}\|\iota_{Y_{\mathrm{d}\phi}}\mathcal{L}_X g - \mathrm{div}X\cdot\mathrm{d}\phi\|_{W^{1,p^*}}\\ &\le \big(A + B + C\|\alpha\|_{(W^{1,p^*})^*}\big)\|\phi\|_{W^{2,p^*}}\end{aligned}$$

Note that here the constant C is varying continuously with the metric g and the vector field X in the $W^{2,\infty}$-topology.

Now if $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$, then the above calculations show in the cases $k = 0$ and $k = -1$ that $\alpha(X)$ is a solution of the weak Neumann problem (wNP) for some $f \in \big(W^{1-k,p^*}(M)\big)^*$ with $\|f\|_{(W^{1-k,p^*})^*} \le A + B + C\|\alpha\|_{(W^{-k,p^*})^*}$. (Here we identify $W^{-1,p}(\cdot) = (W^{1,p^*}(\cdot))^*$ and $W^{-2,p}(\cdot) = (W^{2,p^*}(\cdot))^*$.) So the regularity theorem 2.3′ asserts that $\alpha(X) \in W^{k+1,p}(M)$ with the estimate

$$\begin{aligned}\|\alpha(X)\|_{W^{k+1,p}} &\leq C\big(\|f\|_{(W^{1-k,p^*})^*} + \big|\langle\, \alpha(X)\,,\, 1\,\rangle\big|\big) \\ &\leq C\big(A + B + \|\alpha\|_{(W^{-k,p^*})^*}\big).\end{aligned}$$

Here again C denotes any constant, and those constants depend continuously in the on the metric and the vector field with respect to the $W^{1,\infty}$-topology in case (ii) and the $W^{2,\infty}$-topology in case (iii). The dependence on the vector field and the metric also comes in when we estimate

$$\big|\langle\, \alpha(X)\,,\, 1\,\rangle\big| \;\leq\; \|\alpha(X)\|_{(W^{-k,p^*})^*}\|1\|_{W^{-k,p^*}} \;\leq\; C\|\alpha\|_{(W^{-k,p^*})^*}.$$

The case $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$ works exactly the same way, just that the regularity theorem D.2′ for the weak Dirichlet problem directly gives

$$\|\alpha(X)\|_{W^{k+1,p}} \;\leq\; C\|f\|_{(W^{1-k,p^*})^*} \;\leq\; C\big(A + B + \|\alpha\|_{(W^{-k,p^*})^*}\big).$$

□

With theorem 5.3 established, the further results in this chapter are easy corollaries: We first prove theorem 5.5, where the choice of the test function space $\mathcal{T}$ is determined by the vector field X in order to make certain boundary terms vanish. From this we deduce theorem 5.2, making precise the argument that was lined out on page 78 : For different components, i.e. choices of the vector field X the regularity result uses the test function space $\mathcal{C}^\infty_\delta(M)$, i.e. the Dirichlet problem, or the test function space $\mathcal{C}^\infty_\nu(M)$, i.e. the (inhomogeneous) Neumann problem. Finally, we also show how theorem 5.1 and thus theorem D follows from theorem 5.3 by choices of X and $\mathcal{T}$ that make those boundary terms vanish that do not already vanish due to the boundary condition $*\alpha|_{\partial M} = 0$ or $\alpha|_{\partial M} = 0$.

Proof of theorem 5.5 :
Fix a smooth vector field X that is either perpendicular or tangential to the boundary ∂M. This induces a choice of test function space $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$ or $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$. In order to apply theorem 5.3 we establish the following weak equations for all $\phi \in \mathcal{T}$: Firstly,

$$\begin{aligned}\int_M \langle\, \alpha\,,\, \mathrm{d}(\mathcal{L}_X\phi)\,\rangle &= \int_M \langle\, \mathrm{d}^*\alpha\,,\, \mathcal{L}_X\phi\,\rangle + \int_{\partial M} \mathcal{L}_X\phi \;*\alpha \\ &= \int_M \langle\, G\,,\, \mathcal{L}_X\phi\,\rangle \\ &= \int_M \langle\, -\mathcal{L}_X G - \mathrm{div}X \cdot G\,,\, \phi\,\rangle + \int_{\partial M} g(X,\nu)\cdot G\cdot\phi \\ &= \int_M f_1\cdot\phi.\end{aligned}$$

The first boundary term vanishes since $*\alpha|_{\partial M} = 0$ on $N \subset \partial M$ and $\mathcal{L}_X\phi = 0$ on $\partial M \setminus N$. The second boundary term vanishes since on ∂M either $g(X,\nu) = 0$ (when X is tangential to ∂M) or $\phi = 0$ (since $\phi \in \mathcal{C}^\infty_\delta(M)$ if X is perpendicular to ∂M). So we have

$$f_1 \;=\; -\mathcal{L}_X G - \mathrm{div} X \cdot G \;\in\; W^{k-1,p}(M).$$

Secondly, we obtain for all $\phi \in \mathcal{T}$

$$\begin{aligned}
&\int_M \langle\, \alpha \,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi)\,\rangle \\
&= -\int_M \langle\, \mathrm{d}\alpha \,,\, \mathrm{d}(\phi\cdot\iota_X g)\,\rangle + \int_M \langle\, \mathrm{d}\alpha \,,\, \phi\cdot \mathrm{d}\iota_X g\,\rangle - \int_{\partial M} \alpha\wedge *(\iota_X g\wedge \mathrm{d}\phi) \\
&= -\int_M \langle\, \gamma \,,\, \phi\cdot\iota_X g\,\rangle - \int_{\partial M} \phi\cdot i_X g\wedge *\omega + \int_M \langle\, \mathrm{d}\alpha \,,\, \phi\cdot \mathrm{d}\iota_X g\,\rangle \\
&= \int_M \bigl(-\langle\, \gamma \,,\, \iota_X g\,\rangle + \langle\, \mathrm{d}\alpha \,,\, \mathrm{d}\iota_X g\,\rangle\bigr)\cdot\phi - \int_{\partial M} \phi\cdot i_X g\wedge *\omega \\
&= \int_M f_2\cdot\phi \;+ \int_{\partial M} h\cdot\phi .
\end{aligned}$$

To see that the first boundary term vanishes consider local coordinates near ∂M in which $\{x_0 = 0\}$ describes ∂M and $\frac{\partial}{\partial x^0}$ is the unit normal ν. In the case when $X = \psi\cdot\nu$ and $\phi \in \mathcal{C}^\infty_\delta(M)$, one then has $\iota_X g\wedge \mathrm{d}\phi \;=\; \psi \mathrm{d}x^0 \wedge \frac{\partial\phi}{\partial x^0}\mathrm{d}x^0 \;=\; 0$ on ∂M. When X is tangential to ∂M and $\phi \in \mathcal{C}^\infty_\nu(M)$, then neither $i_X g$ nor $\mathrm{d}\phi$ has a component along $\mathrm{d}x^0$ and hence $*(\iota_X g\wedge \mathrm{d}\phi)|_{\partial M}$ has to vanish. So we find

$$\begin{aligned}
f_2 \;&=\; -\langle\, \gamma \,,\, \iota_X g\,\rangle + \langle\, \mathrm{d}\alpha \,,\, \mathrm{d}\iota_X g\,\rangle \;\in\; W^{k-1,p}(M), \\
h \;&=\; *\bigl(i_X g\wedge *\omega\;|_{\partial M}\bigr) \;\in\; W^{k,p}_\delta(M).
\end{aligned}$$

Now theorem 5.3 (i) asserts that $\alpha(X) \in W^{k+1,p}(M)$ and

$$\begin{aligned}
\|\alpha(X)\|_{W^{k+1,p}} &\le C\bigl(\|\mathcal{L}_X G + \mathrm{div}X\cdot G\|_{W^{k-1,p}} + \|\,\langle\, \gamma \,,\, \iota_X g\,\rangle - \langle\, \mathrm{d}\alpha \,,\, \mathrm{d}\iota_X g\,\rangle\,\|_{W^{k-1,p}} \\
&\qquad + \|\,*\,\bigl(i_X g\wedge *\omega\;|_{\partial M}\bigr)\|_{W^{k,p}_\delta} + \|\alpha\|_{W^{k,p}}\bigr) \\
&\le C\bigl(\|G\|_{W^{k,p}} + \|\gamma\|_{W^{k-1,p}} + \|\omega\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}\bigr).
\end{aligned}$$

Here the first constant from theorem 5.3 has the claimed $W^{k+1,\infty}$-dependence on the metric and vector field, and in the second estimate we pick up further derivatives of X and g, but these are only of order k. □

Proof of theorem 5.2 :
By remark B.1 it suffices to prove the regularity and estimate for $\alpha(X)$ for all smooth vector fields $X \in \Gamma(\mathrm{T}M)$ that coincide with a coordinate vector field on some coordinate patch $U \subset M$ of a fixed atlas. The coordinate charts that intersect

the boundary ∂M can be chosen such that ∂M is mapped to the hyperplane $\{x^0 = 0\}$ and the coordinate vector field $\frac{\partial}{\partial x^0}$ restricts to the unit normal ν on ∂M. [1] Now the coordinate vector fields are either perpendicular or tangential to the boundary ∂M, and these properties are preserved under multiplication with a cutoff function. Hence it suffices to consider vector fields $X \in \Gamma(\mathrm{T}M)$ such that either $X|_{\partial M} = \psi \cdot \nu$ for some function $\psi \in \mathcal{C}^\infty(\partial M)$ or $X|_{\partial M} \in \Gamma(\mathrm{T}\partial M)$. Thus we have reduced this theorem to theorem 5.5.

The constant in the estimate of theorem 5.5 also depends continuously on the metric and the vector field X with respect to the $W^{k+1,\infty}$-topology. So it remains to consider the constant in the equivalence of the usual Sobolev norm on 1-forms and the norm defined via coordinate charts in remark B.1. If we fix one metric g and consider another metric h then we can use the same coordinate vector fields except for the normal direction near the boundary. But since the unit normals are related by $\nu_h^i = h^{ij} g_{jk} \nu_g^k$ in local coordinates the corresponding vector fields X are $W^{k+1,\infty}$-close if h lies in a suitable $W^{k+1,\infty}$-neighbourhood of g. Moreover, the constant in the equivalence of the $W^{k+1,p}$-, $W^{k,p}$-, and $W^{k-1,p}$-norms depends continuously on the metric and coordinate vector fiels with respect to the $W^{k+1,\infty}$-topology. Together this gives the claimed continuity. □

Proof of theorem 5.1 :
As in the previous proof, it suffices to establish the estimate for $A(X)$ for smooth vector fields $X \in \Gamma(\mathrm{T}M)$ that are either perpendicular or tangential to the boundary ∂M. (This spans all coordinate vector fields for an appropriate choice of an atlas.) Also as previously, one can choose the relevant vector fields to vary continuously with the metric with respect to any $W^{k,\infty}$-topology on both. In order to deduce the claimed estimate on $A(X)$ from theorem 5.3 we need the following weak equations: Firstly,

$$\begin{aligned}\int_M \langle\, A \,,\, \mathrm{d}(\mathcal{L}_X \phi)\, \rangle &= \int_M \langle\, \mathrm{d}^* A \,,\, \mathcal{L}_X \phi \,\rangle + \int_{\partial M} \mathcal{L}_X \phi \, * A \\ &\leq \|\mathrm{d}^* A\|_p \|\mathcal{L}_X \phi\|_{p^*} \;\leq\; C \|\mathrm{d}^* A\|_p \|\phi\|_{W^{1,p^*}} .\end{aligned}$$

If $*A|_{\partial M} = 0$ then this holds for all $\phi \in \mathcal{C}^\infty(M)$ since the boundary term vanishes. Otherwise it holds if we make sure that $\mathcal{L}_X \phi|_{\partial M} = 0$. So in the case $A|_{\partial M} = 0$ we choose $\phi \in \mathcal{C}^\infty_\nu(M)$ if X is perpendicular to ∂M, and we choose $\phi \in \mathcal{C}^\infty_\delta(M)$ if X is tangential to ∂M. The constant C then depends on the L^∞-norm of X and thus varies continuously with the metric in the L^∞-topology. Secondly, we obtain

$$\begin{aligned}\int_M \langle\, A \,,\, \mathrm{d}^*(\iota_X g \wedge \mathrm{d}\phi)\, \rangle &= \int_M \langle\, \mathrm{d}A \,,\, \iota_X g \wedge \mathrm{d}\phi \,\rangle - \int_{\partial M} A \wedge *(\iota_X g \wedge \mathrm{d}\phi) \\ &\leq \|\mathrm{d}A\|_p \|i_X g \wedge \mathrm{d}\phi\|_{p^*} \;\leq\; C \|\mathrm{d}A\|_p \|\phi\|_{W^{1,p^*}} .\end{aligned}$$

[1] In order to achieve this one extends ν to a vector field on M. Then its flow defines a tubular neighbourhood $(-\varepsilon, 0] \times \partial M \to M$, whose first component can be combined with a coordinate chart of ∂M to give the required coordinates on M.

Now the boundary term vanishes and this is true for all $\phi \in \mathcal{C}^\infty(M)$ if we assume $A|_{\partial M} = 0$. In the case $*A|_{\partial M} = 0$ we have to make appropriate choices of the test function space: If X is perpendicular to the boundary, then the estimate holds for all $\phi \in \mathcal{C}^\infty_\nu(M)$; if X is tangential, then it holds for $\phi \in \mathcal{C}^\infty_\delta(M)$. This is since $*(\iota_X g \wedge \mathrm{d}\phi)|_{\partial M}$ vanishes as shown before in the proof of theorem 5.5. Again, the constant C varies continuously with the metric in the L^∞-topology. So theorem 5.3 (ii) applies in the following cases:

- If $*A|_{\partial M} = 0$: Either X is perpendicular to ∂M and $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$ or X is tangential to ∂M and $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$.
- If $A|_{\partial M} = 0$: Either X is perpendicular to ∂M and $\mathcal{T} = \mathcal{C}^\infty_\nu(M)$ or X is tangential to ∂M and $\mathcal{T} = \mathcal{C}^\infty_\delta(M)$.

In all four cases we obtain

$$\|A(X)\|_{W^{1,p}} \leq C\big(\|\mathrm{d}^*A\|_p + \|\mathrm{d}A\|_p + \|A\|_p\big)$$

and thus with a constant C that varies $W^{1,\infty}$-continuously with the metric

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}^*A\|_p + \|\mathrm{d}A\|_p + \|A\|_p\big).$$

This proves (i). In order to prove the estimate in (ii) we assume $H^1(M;\mathbb{R}) = 0$ and consider the following Banach spaces:

$$\begin{aligned} X &= \big\{A \in W^{1,p}(M, \mathrm{T}^*M) \;\big|\; *A|_{\partial M} = 0\big\}, \\ Y &= L^p(M, \Lambda^2\mathrm{T}^*M) \oplus L^p(M), \\ Z &= L^p(M, \mathrm{T}^*M). \end{aligned}$$

X is indeed a Banach space since by theorem B.10 the restriction to ∂M induces a continuous map $W^{1,p}(M, \Lambda^{n-1}\mathrm{T}^*M) \to L^1(M, \Lambda^{n-1}\mathrm{T}^*\partial M)$. Moreover, due to the Sobolev embedding theorem B.2 the inclusion $K : X \to Z$ is compact. We equip Y with the metric $\|(\omega, f)\|_Y = \|\omega\|_p + \|f\|_p$, then the operator $D := \mathrm{d} \oplus \mathrm{d}^* : X \to Y$ obviously is linear and continuous. The already established estimate (i) can be rephrased as

$$\|A\|_X \leq C\big(\|DA\|_Y + \|KA\|_Z\big)$$

for all $A \in X$ and some constant C. Now we just have to check that D is injective, then lemma E.3 implies the claimed estimate $\|A\|_X \leq C\|DA\|_Y$ for all $A \in X$ and some constant C. To check the injectivity of D consider an element $A \in W^{1,p}(M, \mathrm{T}^*M)$ in the kernel, i.e. with $\mathrm{d}A = 0$ and $\mathrm{d}^*A = 0$. We can iteratively apply theorem 5.2 as above with $G = 0$, $\gamma = 0$, and $\omega = 0$ to see that A is smooth on M. Now $H^1(M;\mathbb{R}) = 0$ implies that $A = \mathrm{d}f$ for some function $f \in \mathcal{C}^\infty(M)$. Since $*A|_{\partial M} = 0$ this function solves the homogeneous Neumann problem $\Delta f = \mathrm{d}^*A = 0$ and $\frac{\partial f}{\partial \nu} = A(\nu) = 0$. Thus corollary 1.9 asserts that f

equals a constant and hence $A = 0$. This proves the injectivity of the operator D and establishes

$$\|A\|_{W^{1,p}} \;\leq\; C\|DA\|_Y \;=\; C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p\big).$$

However, the open mapping theorem does not give us a control of the constant in this estimate when the metric varies. (Note that this in particular changes the domain X of the operator D.) We take this opportunity to exemplify how to prove the continuity of constants in detail. In the following, the subscript g or h indicates the metric with respect to which an operator or a norm is defined.

Fix $\lambda > 1$, a metric g on M, and let C_g be the optimal constant in the estimate (ii) when using the metric g. The claim is that $\lambda^{-1}C_g \leq C_h \leq \lambda C_g$ for all other metrics h with optimal constant C_h and $\|h-g\|_{g,W^{1,\infty}} \leq \delta$ for sufficiently small $\delta > 0$ (depending on g).

Firstly, the $W^{1,p}$-norm on 1-forms only depends on the metric, its inverse, and its Christoffel symbols. So if δ is small enough then for all $A \in W^{1,p}(M, \mathrm{T}^*M)$

$$\lambda^{-\frac{1}{4}}\|A\|_{h,W^{1,p}} \;\leq\; \|A\|_{g,W^{1,p}} \;\leq\; \lambda^{\frac{1}{4}}\|A\|_{h,W^{1,p}} \tag{5.5}$$

and the same equivalence also holds for the L^p-norms of 2-forms and functions (i.e. $\mathrm{d}A$ and d^*A). But now the estimate for the metric g does not apply to every $A \in W^{1,p}(M, \mathrm{T}^*M)$ with $*_h A|_{\partial M} = 0$ since this does not necessarily meet the boundary condition with respect to g. However, the dual vector field X^h_A to A defined by $\iota_{X_A} h = A$ is tangential to ∂M, so $\tilde{A} := \iota_{X^h_A} g$ meets the boundary condition $*_g\tilde{A}|_{\partial M} = 0$ with respect to g . Note that the dual vector field for $\tilde{A}$ with respect to the metric g is $X^g_{\tilde{A}} = X^h_A$. So there is a one-to-one correspondence $\tilde{A} = \iota_{X^g_{\tilde{A}}} g \mapsto \iota_{X^h_A} h = A$ between the boundary value spaces with respect to the metrics g and h. Locally $A_i = h_{ij}g^{jk}\tilde{A}_k$, so

$$\|\tilde{A} - A\|_{g,W^{1,p}} \leq \|g^{-1}\|_{g,W^{1,\infty}}\|g-h\|_{g,W^{1,\infty}}\|\tilde{A}\|_{g,W^{1,p}}. \tag{5.6}$$

Now consider the difference between $\mathrm{d}^*_g\tilde{A}$ and $\mathrm{d}^*_h A$. Again from the local representation one sees that it is bounded by $o(\delta)\big(|\tilde{A}| + |\mathrm{d}\tilde{A}|\big)$, where o is a continuous function with $o(0) = 0$. So for small enough $\delta > 0$

$$\|\mathrm{d}^*_g\tilde{A} - \mathrm{d}^*_h A\|_{g,L^p} \leq \tfrac{1-\lambda^{-\frac{1}{4}}}{2\lambda^2 C_g}\|\tilde{A}\|_{g,W^{1,p}}. \tag{5.7}$$

Also choose $\delta > 0$ such that in (5.6)

$$\|\tilde{A} - A\|_{g,W^{1,p}} \leq \tfrac{1-\lambda^{-\frac{1}{4}}}{2\lambda^2 C_g}\|\tilde{A}\|_{g,W^{1,p}}. \tag{5.8}$$

We now establish $C_h \leq \lambda C_g$ by showing that estimate (ii) holds with the constant λC_g for all metrics h with sufficiently small $\|h-g\|_{g,W^{1,\infty}} \leq \delta$. Consider any

$A \in W^{1,p}(M, \mathrm{T}^*M)$ with $*_h A|_{\partial M} = 0$ and the corresponding $\tilde{A} = \iota_{X^h_A} g$. Use (5.8) in the estimate (ii) on $\tilde{A}$ with the metric g to obtain

$$\begin{aligned}\|\tilde{A}\|_{g,W^{1,p}} &\leq C_g\big(\|\mathrm{d}\tilde{A}\|_{g,L^p} + \|\mathrm{d}^*_g\tilde{A}\|_{g,L^p}\big) \\ &\leq C_g\big(\|\mathrm{d}A\|_{g,L^p} + \|\tilde{A}-A\|_{g,W^{1,p}} + \|\mathrm{d}^*_h A\|_{g,L^p} + \|\mathrm{d}^*_g\tilde{A} - \mathrm{d}^*_h A\|_{g,L^p}\big) \\ &\leq C_g\big(\|\mathrm{d}A\|_{g,L^p} + \|\mathrm{d}^*_h A\|_{g,L^p}\big) + (1-\lambda^{-\frac14})\|\tilde{A}\|_{g,W^{1,p}}.\end{aligned}$$

Now the last term can be taken to the left hand side, then (5.5) gives

$$\begin{aligned}\|\tilde{A}\|_{g,W^{1,p}} &\leq \lambda^{\frac14} C_g\big(\|\mathrm{d}A\|_{g,L^p} + \|\mathrm{d}^*_h A\|_{g,L^p}\big) \\ &\leq \lambda^{\frac12} C_g\big(\|\mathrm{d}A\|_{h,L^p} + \|\mathrm{d}^*_h A\|_{h,L^p}\big).\end{aligned}$$

Moreover, in (5.6) choose $\delta > 0$ sufficiently small to obtain

$$\|A\|_{g,W^{1,p}} \;\leq\; \|\tilde{A}\|_{g,W^{1,p}} + \|\tilde{A}-A\|_{g,W^{1,p}} \;\leq\; \lambda^{\frac14}\|\tilde{A}\|_{g,W^{1,p}}.$$

The last two inequalities together with (5.5) then give the result:

$$\|A\|_{h,W^{1,p}} \;\leq\; \lambda^{\frac14}\|A\|_{g,W^{1,p}} \;\leq\; \lambda C_g\big(\|\mathrm{d}A\|_{h,L^p} + \|\mathrm{d}^*_h A\|_{h,L^p}\big).$$

Conversely, in order to establish $C_g \leq \lambda C_h$ consider any $\tilde{A} \in W^{1,p}(M, \mathrm{T}^*M)$ with $*_g\tilde{A}|_{\partial M} = 0$ and the corresponding $A = \iota_{X^g_{\tilde{A}}} h$. Firstly, $\delta > 0$ can be chosen sufficiently small such that (5.6) gives

$$\|\tilde{A}\|_{g,W^{1,p}} \;\leq\; \|A\|_{g,W^{1,p}} + \|\tilde{A}-A\|_{g,W^{1,p}} \;\leq\; \|A\|_{g,W^{1,p}} + (1-\lambda^{-\frac14})\|\tilde{A}\|_{g,W^{1,p}}$$

and thus $\|\tilde{A}\|_{g,W^{1,p}} \leq \lambda^{\frac14}\|A\|_{g,W^{1,p}}$. Then similarly as above the estimate (ii) on A with the metric h yields

$$\begin{aligned}&\|\tilde{A}\|_{g,W^{1,p}} \\ &\leq \lambda^{\frac12}\|A\|_{h,W^{1,p}} \\ &\leq \lambda^{\frac12} C_h\big(\|\mathrm{d}A\|_{h,L^p} + \|\mathrm{d}^*_h A\|_{h,L^p}\big) \\ &\leq \lambda^{\frac34} C_h\big(\|\mathrm{d}\tilde{A}\|_{g,L^p} + \|\mathrm{d}^*_g\tilde{A}\|_{g,L^p} + \|A-\tilde{A}\|_{g,W^{1,p}} + \|\mathrm{d}^*_h A - \mathrm{d}^*_g\tilde{A}\|_{g,L^p}\big) \\ &\leq \lambda^{\frac34} C_h\big(\|\mathrm{d}\tilde{A}\|_{g,L^p} + \|\mathrm{d}^*_g\tilde{A}\|_{g,L^p}\big) + (1-\lambda^{\frac14})\|\tilde{A}\|_{g,W^{1,p}}.\end{aligned}$$

Here we have used (5.5), (5.7), and (5.8) as well as $C_h \leq \lambda C_g$, which was established before. Finally, this implies

$$\|\tilde{A}\|_{g,W^{1,p}} \leq \lambda C_h\big(\|\mathrm{d}\tilde{A}\|_{g,L^p} + \|\mathrm{d}^*_g\tilde{A}\|_{g,L^p}\big),$$

which finishes the proof of the continuity of the constants and hence of the whole theorem. □

Chapter 6

Uhlenbeck Gauge

In this chapter we consider a trivialized chart of a principal bundle and prove the existence of a Coulomb-type gauge (the 'Uhlenbeck gauge') for connections with small energy. This is theorem B. These gauges will then be patched up in chapter 7 to prove the weak compactness.

Theorem 6.1 (Uhlenbeck Gauge)
Let M be a Riemannian n-manifold and let G be a compact Lie group. Suppose that $1 < q \leq p < \infty$ such that $q \geq \frac{n}{2}$, $p > \frac{n}{2}$, and in case $q < n$ assume $p \leq \frac{nq}{n-q}$. Then there exist constants C_{Uh} and $\varepsilon_{Uh} > 0$ such that the following holds:

For every point in M one can find a neighbourhood $U \subset M$ such that for every connection $A \in \mathcal{A}^{1,p}(U)$ with $\mathcal{E}(A) \leq \varepsilon_{Uh}$ there exists a gauge transformation $u \in \mathcal{G}^{2,p}(U)$ such that

(i) $\mathrm{d}^*(u^*A) = 0$, *(iii)* $\|u^*A\|_{W^{1,q}} \leq C_{Uh}\|F_A\|_q$,

(ii) $*(u^*A)|_{\partial U} = 0$, *(iv)* $\|u^*A\|_{W^{1,p}} \leq C_{Uh}\|F_A\|_p$.

Here for every local trivialization $P|_U \xrightarrow{\sim} U \times G$ of a principal G-bundle P the space $\mathcal{A}^{1,p}(U) = W^{1,p}(U, \mathrm{T}^*U \otimes \mathfrak{g})$ represents the connections on $P|_U$. The energy of a connection $A \in \mathcal{A}^{1,p}(U)$ is given by

$$\mathcal{E}(A) = \int_U |F_A|^q = \|F_A\|_q^q,$$

and $\mathcal{G}^{2,p}(U) = W^{2,p}(U, G)$ represents the gauge transformations on $P|_U$.

Remark 6.2

a) The assumption $p > \frac{n}{2}$ ensures that $\mathcal{G}^{2,p}(M)$ is a well defined group and acts continuously on $\mathcal{A}^{1,p}(M)$, see lemma A.6.

   The theorem actually extends to the case $q = p = \frac{n}{2}$ for $n \geq 3$ by a weak limit argument that we explain at the end of this chapter. Here $W^{2,\frac{n}{2}}(M,G)$ is defined via a fixed embedding $G \hookrightarrow \mathbb{R}^m$, e.g. as a matrix group.

b) Property (i) is the Coulomb gauge condition. Condition (ii) is well posed since $W^{1,p}(U) \hookrightarrow L^p(\partial U)$ by theorem B.10.

c) We say that a connection $A \in \mathcal{A}^{1,p}(U)$ is in **Uhlenbeck gauge** if it satisfies

$$\text{(i)}\ \ \mathrm{d}^* A = 0, \qquad \text{(iii)}\ \ \|A\|_{W^{1,q}} \leq C_{Uh}\|F_A\|_q$$
$$\text{(ii)}\ \ *A|_{\partial U} = 0, \qquad \text{(iv)}\ \ \|A\|_{W^{1,p}} \leq C_{Uh}\|F_A\|_p.$$

Thus the statement of the theorem is that u^*A is in Uhlenbeck gauge (recall $|F_{u^*A}| = |F_A|$).

d) For $q = \frac{n}{2}$ and $\frac{n}{2} \leq p \leq n$ this is the original version of Uhlenbeck's theorem [U2, Thm.2.1]. The more general formulation was inspired by the fact that for $n = 2$ control of the L^1-norm of the curvature does not seem to suffice in order to obtain the result. Problems arise in the proof of the crucial $W^{1,p}$-estimate since the Calderon-Zygmund inequality does not hold for $p = 1$.

Moreover, it seems that in order to obtain a $W^{1,p}$-control in (iv) for $p > n$, one needs small energy for $q > \frac{n}{2}$.

e) The domains U will be geodesic balls or (in the case of the given point lying on the boundary of M) “eggs” as defined later. The radius of these domains can be chosen arbitrarily small without affecting the constants C_{Uh} and ε_{Uh}.

That way the neighbourhood U of the given point can be made to lie within any other fixed neighbourhood $\tilde{U}$ of that point.

All the rest of this chapter will be concerned with the proof of this theorem. Let M be equipped with a smooth metric g, then we first reduce the general setting to two model cases according to wether the given point $x \in M$ lies in the interior $\mathrm{int}(M)$ or on the boundary ∂M of the manifold. In both cases, for every given $\delta > 0$, we find a chart $\psi_\sigma : B \to M$ from a fixed domain $B \subset \mathbb{R}^n$ to a neighbourhood U of x such that $\|\sigma^{-2}\psi_\sigma^* g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$ for some $\sigma \in (0, 1]$.

$x \in \mathrm{int}(M)$:
Let $B \subset \mathbb{R}^n$ be the open unit ball centered at 0 and let $\delta > 0$ be given. Then for small enough $\sigma \in (0, 1]$ there exists a normal coordinate chart $\psi : \sigma B \to M$ at $\psi(0) = x$, that is $\psi^* g(0) = \mathbb{1}$ with vanishing first derivatives. Then for possibly even smaller $\sigma > 0$ the chart restricted to σB meets $\|\psi^* g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$ for any a priori fixed $\delta > 0$. Now consider the chart $\psi_\sigma : B \to M$, $z \mapsto \psi(\sigma z)$ to a neighbourhood $U = \psi(\sigma B)$ of x. The pullback metric $\psi_\sigma^* g(z) = \sigma^2 \psi^* g(\sigma z)$ is no longer close to the identity, but $\sigma^{-2}\psi_\sigma^* g$ is $W^{2,\infty}$-close to $\mathbb{1}$. Indeed, by the choice of $\sigma \in (0, 1]$ the bound on the first derivatives is preserved and even the second derivatives $\nabla^2(\sigma^{-2}\psi_\sigma^* g)(z) = \sigma^2 \nabla^2(\psi^* g)(\sigma z)$ can be made as small as we wish. This is due to the fact that $\psi^* g$ is smooth on the closure of σB and thus all its derivatives can be bounded.

<u>***$x \in \partial M$***</u>:
Choose coordinates near x such that 0 maps to x and ∂M corresponds to the boundary of the half space $\mathbb{H}^n = \{(z_0, \dots, z_{n-1}) \in \mathbb{R}^n \mid z_0 \geq 0\}$. Note that these coordinates can not in addition be chosen normal at x. However, the chart $\psi : V \to M$ on a relatively open subset $V \subset \mathbb{H}^n$ can be chosen such that $\psi^* g(0) = \mathbb{1}$. So if V is chosen sufficiently small then $\|\psi^* g - \mathbb{1}\|_\infty \leq \delta$ for any a priori fixed $\delta > 0$.

Now let $B \subset \{z \in \mathbb{H}^n \mid |z| \leq 1\}$ be an "egg squeezed to the boundary" as in figure 4.1, i.e. an open subset of $\mathbb{H}^n$ that contains a neighbourhood of $0 \in \partial\mathbb{H}^n$, is starshaped with respect to 0, and has a smooth boundary. Then σB lies within V for sufficiently small $\sigma > 0$ and hence $\psi_\sigma : B \to M$, $z \mapsto \psi(\sigma z)$ is a well defined coordinate chart of a neighbourhood $U = \psi(\sigma B) \subset M$ of x. Again consider the rescaled metric $\sigma^{-2}\psi_\sigma^* g(z) = \psi^* g(\sigma z)$ for $\sigma \in (0, 1]$.

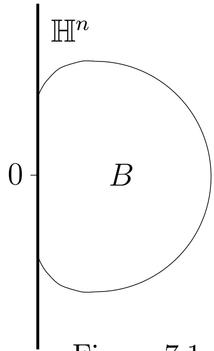

Figure 7.1: The "egg"

As before $\nabla(\psi^* g)$ is smooth and thus bounded on σV, so the first derivatives $\nabla(\sigma^{-2}\psi_\sigma^* g)(z) = \sigma\nabla(\psi^* g)(\sigma z)$ can be made small by the choice of $\sigma > 0$. The same works for the second derivatives, and we obtain $\|\sigma^{-2}\psi_\sigma^* g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$ for some $\sigma \in (0, 1)$.

So by working in special local coordinates it suffices to prove the following model case version of the Uhlenbeck gauge theorem and then examine the effect of rescaling the metric.

Theorem 6.3 (Uhlenbeck gauge theorem for the model cases)
Let G be a compact Lie group and let $B \subset \mathbb{R}^n$ be the 1-ball or "egg" as above. Let $1 < q \leq p < \infty$ be such that $q \geq \frac{n}{2}$, $p > \frac{n}{2}$, and in case $q < n$ assume $p \leq \frac{nq}{n-q}$. Then there exist constants $\delta > 0$, C_{Uh}, and $\varepsilon_{Uh} > 0$ such that the following holds:

If B is equipped with a smooth metric g such that $\|g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$ then for every connection $A \in \mathcal{A}^{1,p}(B)$ with $\mathcal{E}(A) \leq \varepsilon_{Uh}$ there exists a gauge transformation $u \in \mathcal{G}^{2,p}(B)$ such that u^*A is in Uhlenbeck gauge gauge with respect to the metric g :

(i) $\mathrm{d}^*(u^*A) = 0$, *(iii)* $\|u^*A\|_{W^{1,q}} \leq C_{Uh}\|F_A\|_q$
(ii) $*(u^*A)|_{\partial B} = 0$, *(iv)* $\|u^*A\|_{W^{1,p}} \leq C_{Uh}\|F_A\|_p$.

Remark 6.4 *In this theorem it actually suffices to have $W^{1,\infty}$-control of the metric, i.e. $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$; see the footnote on page 101.*

Let us first establish some special Sobolev product estimates that will be used several times in the proof. These are the reason for the assumption $p \leq \frac{nq}{n-q}$ in case $q < n$.

Lemma 6.5 *Let M be a compact Riemannian n-manifold. Let $1 \le q \le p < \infty$ be such that $q \ge \frac{n}{2}$, and in case $q < n$ assume $p \le \frac{nq}{n-q}$. Then there exists a constant C_S such that for all $f, g \in W^{1,p}(M)$*

$$\|f \cdot g\|_q \le C_S \|f\|_r \|g\|_{W^{1,q}},$$
$$\|f \cdot g\|_p \le C_S \|f\|_r \|g\|_{W^{1,p}},$$
$$\|f \cdot g\|_p \le C_S \|f\|_{W^{1,q}} \|g\|_{W^{1,p}}.$$

Here $r = \frac{nq}{n-q}$ if $q < n$, $r = 2p$ if $q = n$, and $r = \infty$ if $q > n$.

Proof: We prove the first two estimates for $s = q$ and $s = p$ at the same time, then the third follows from the second.

First consider the case $q < n$. Use the Hölder inequality with $\frac{1}{\tau} = \frac{1}{s} - \frac{n-q}{nq}$ and the Sobolev inequality for $W^{1,s} \hookrightarrow L^\tau$ to obtain a constant C such that

$$\|f \cdot g\|_s \;\le\; \|f\|_{\frac{nq}{n-q}} \|g\|_\tau \;\le\; C \|f\|_r \|g\|_{W^{1,s}}.$$

For the Hölder inequality note that due to $q \le p \le \frac{nq}{n-q}$ in both cases $s \le \frac{nq}{n-q}$ and $\frac{1}{s} - \frac{n-q}{nq} \le \frac{1}{n}$, so $\tau \in [n, \infty]$. The condition for the Sobolev inequality is $1 - \frac{n}{s} \ge -\frac{n}{\tau} = -\frac{n}{s} + \frac{n-q}{q}$, which is satisfied due to $q \ge \frac{n}{2}$.

In case $q = n$ the Sobolev inequality for $W^{1,s} \hookrightarrow L^{2s}$ can be applied due to $s \ge \frac{n}{2}$ and it gives

$$\|f \cdot g\|_s \;\le\; \|f\|_{2s} \|g\|_{2s} \;\le\; C \|f\|_r \|g\|_{W^{1,s}}.$$

Here the constant C also contains the constant C' from $\|f\|_{2s} \le C' \|f\|_r$, which is due to $2q \le 2p = r$ and the finite volume of B.

In the case $q > n$ we simply use the Sobolev inequality for $W^{1,s} \hookrightarrow L^s$,

$$\|f \cdot g\|_s \;\le\; \|f\|_\infty \|g\|_s \;\le\; C \|f\|_r \|g\|_{W^{1,s}}.$$

Finally, the third estimate follows in all three cases from the Sobolev inequality for $W^{1,q} \hookrightarrow L^r$, for which r was chosen appropriately. □

Proof of theorem 6.3 :
Define a modified energy

$$\mathcal{E}'(A) = \int_B |F_A(x)|^q \, \mathrm{d}^n x$$

in which $|F_A|$ is calculated with respect to the metric g on B but integration is with respect to the Euclidean volume form $\mathrm{d}^n x$. Then let

$$\mathcal{A}_\varepsilon := \left\{ A \in \mathcal{A}^{1,p}(B) \;\middle|\; \mathcal{E}'(A) \le \varepsilon \right\}$$

be equipped with the $W^{1,p}$-topology and consider its subset $\mathcal{S}_\varepsilon$ that consists of all $A \in \mathcal{A}_\varepsilon$ for which a gauge transformation $u \in \mathcal{G}^{2,p}(B)$ exists such that u^*A is in Uhlenbeck gauge.

We will prove by continuous induction that $\mathcal{S}_\varepsilon = \mathcal{A}_\varepsilon$ for some C_{Uh}, $\varepsilon > 0$, and all metrics with sufficiently small $\delta > 0$ in $\|g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$. If instead of $\mathcal{E}'$ the original energy $\mathcal{E}$ was be used in the definition of $\mathcal{A}_\varepsilon$ then this would prove the theorem. But unfortunately we cannot prove the connectedness of $\mathcal{A}_\varepsilon$ with the original energy $\mathcal{E}$. One can, however, choose $\delta > 0$ sufficiently small to obtain $\frac{1}{2} \leq \sqrt{\det g} \leq 2$ and thus

$$\tfrac{1}{2}\mathcal{E}'(A) \leq \mathcal{E}(A) \leq 2\mathcal{E}'(A) \qquad \forall A \in \mathcal{A}^{1,p}(B). \tag{6.1}$$

Hence if $\mathcal{S}_\varepsilon = \mathcal{A}_\varepsilon$ then every connection with energy at most $\varepsilon_{Uh} := \frac{1}{2}\varepsilon$ also lies in $\mathcal{A}_\varepsilon$ and thus can be put into Uhlenbeck gauge, which proves the theorem.

So we have to prove in a first step that for every $\varepsilon > 0$ and every metric the space $\mathcal{A}_\varepsilon$ is connected. As a second step we prove that $\mathcal{S}_\varepsilon$ is closed for every metric and every choice of the constants C_{Uh} and $\varepsilon > 0$. The last and major step is then to find metric independent constants C_{Uh} and $\varepsilon > 0$ such that $\mathcal{S}_\varepsilon \subset \mathcal{A}_\varepsilon$ is open for every metric g with $\|g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$, where δ is chosen appropriately small. When this is established then moreover, the trivial connection $A = 0$ obviously has zero energy and is in Uhlenbeck gauge with respect to every metric, hence $\mathcal{S}_\varepsilon$ is nonempty and thus comprises the whole connected component $\mathcal{A}_\varepsilon$.

1.) Connectedness : *Let $\bar{B}$ be equipped with a smooth metric g and let $\varepsilon > 0$. Then every $A \in \mathcal{A}_\varepsilon$ is connected by a path in $\mathcal{A}_\varepsilon$ to the trivial connection $0 \in \mathcal{A}_\varepsilon$.*

Since B is starshaped with respect to 0 we can define a path $(A_\sigma)_{\sigma\in[0,1]}$ by $A_\sigma(x) = \sigma A(\sigma x)$ for $x \in B$. This path connects 0 to A. Indeed, obviously $A_0 = 0$, $A_1 = A$, and $A_\sigma \in \mathcal{A}^{1,p}(B)$ for all σ. The curvature of the connection A_σ is

$$F_{A_\sigma}(x) = \sigma^2 \mathrm{d}A(\sigma x) + \tfrac{1}{2}\sigma^2[A(\sigma x) \wedge A(\sigma x)] = \sigma^2 F_A(\sigma x)$$

and hence $A_\sigma \in \mathcal{A}_\varepsilon$ for all $\sigma \in [0,1]$ since

$$\begin{aligned}\mathcal{E}'(A_\sigma) &= \int_B \sigma^{2q} \left|F_A(\sigma x)\right|^q \mathrm{d}^n x \\ &= \sigma^{2q-n} \int_{\sigma B} \left|F_A(y)\right|^q \mathrm{d}^n y \\ &\leq \sigma^{2q-n}\mathcal{E}'(A) \ \leq \ \varepsilon.\end{aligned}$$

Here the assumption $q \geq \frac{n}{2}$ comes in crucially and this also is the point that does not work with the original energy $\mathcal{E}$. It remains to show that the path is actually continuous with respect to the $W^{1,p}$-topology. For this purpose we use the Euclidean metric on B to define the norm $\|\cdot\|_{\mathbb{1},W^{1,p}}$ on $\mathcal{A}_\varepsilon$. Since g is a smooth metric on the compact set $\bar{B}$ this norm is equivalent to the $W^{1,p}$-norm with respect to g.

Now for the continuity at $\sigma_0 = 0$ note that $A_0 = 0$ and hence check that $\|A_\sigma\|_{\mathbb{1},W^{1,p}}$ converges to 0 as $\sigma \to 0$. The L^p-norm can be controlled by the Hölder inequality and the Sobolev inequality for $L^{2p} \hookrightarrow W^{1,p}$,

$$\begin{aligned}
\|A_\sigma\|^p_{\mathbb{1},L^p(B)} &= \int_B \sigma^p |A(\sigma x)|^p \mathrm{d}^n x \\
&= \sigma^{p-n} \|A\|^p_{\mathbb{1},L^p(\sigma B)} \\
&\le \sigma^{p-n} \mathrm{Vol}(\sigma B)^{\frac{1}{2}} \|A\|^p_{\mathbb{1},L^{2p}(\sigma B)} \\
&\le \sigma^{p-\frac{n}{2}} \mathrm{Vol}(B)^{\frac{1}{2}} C \|A\|^p_{\mathbb{1},W^{1,p}(B)}.
\end{aligned}$$

This converges to 0 as $\sigma \to 0$ since $p > \frac{n}{2}$. For the derivative term in the Euclidean $W^{1,p}$-norm that is even easier seen:

$$\|\nabla A_\sigma\|^p_{\mathbb{1},L^p(B)} \;=\; \int_B \sigma^{2p} |\nabla A(\sigma x)|^p \mathrm{d}^n x \;=\; \sigma^{2p-n} \|\nabla A\|^p_{\mathbb{1},L^p(\sigma B)} \underset{\sigma \to 0}{\longrightarrow} 0.$$

The continuity at $\sigma_0 > 0$ relies on a sequence $(A^i)_{i\in\mathbb{N}} \subset \mathcal{C}^\infty(\bar{B})$ that converges to A in the $W^{1,p}$-norm. These satisfy $|\nabla A^i(x)| \le C_i$ for all $x \in B$ and some constants C_i. So for all $\sigma \in (0,1]$, $x \in B$, and $i \in \mathbb{N}$

$$\begin{aligned}
|A_\sigma(x) - A_{\sigma_0}(x)| &= |\sigma A(\sigma x) - \sigma_0 A(\sigma_0 x)| \\
&\le |\sigma - \sigma_0| \cdot |A(\sigma_0 x)| + \sigma |A(\sigma x) - A^i(\sigma x)| \\
&\qquad + \sigma |A(\sigma_0 x) - A^i(\sigma_0 x)| + \sigma |A^i(\sigma x) - A^i(\sigma_0 x)| \\
&\le |\sigma - \sigma_0| \cdot |A(\sigma_0 x)| + |A(\sigma x) - A^i(\sigma x)| \\
&\qquad + |A(\sigma_0 x) - A^i(\sigma_0 x)| + |\sigma - \sigma_0| C_i.
\end{aligned}$$

We apply the Euclidean L^p-norm to both sides of this inequality to obtain

$$\begin{aligned}
\|A_\sigma - A_{\sigma_0}\|_{\mathbb{1},L^p} \le |\sigma - \sigma_0| \sigma_0^{-\frac{n}{p}} \|A\|_{\mathbb{1},L^p} + (\sigma^{-\frac{n}{p}} + \sigma_0^{-\frac{n}{p}}) \|A - A^i\|_{\mathbb{1},L^p} \\
+ |\sigma - \sigma_0| \mathrm{Vol}(B)^{\frac{1}{p}} C_i.
\end{aligned}$$

For fixed σ_0 this can be made as small as we wish by the choice of i and $|\sigma - \sigma_0|$: First make sure that $\sigma^{-\frac{n}{p}} + \sigma_0^{-\frac{n}{p}} \le 3\sigma_0^{-\frac{n}{p}}$, then choose i sufficiently large to make the second term small. With this fixed i the first and third term are small by an even smaller choice of $|\sigma - \sigma_0|$. An analogous argument also works for ∇A_σ with bounds C_i' on the second derivatives of the approximating A^i:

$$\begin{aligned}
|\nabla A_\sigma(x) - \nabla A_{\sigma_0}(x)| &= |\sigma^2 \nabla A(\sigma x) - \sigma_0^2 \nabla A(\sigma_0 x)| \\
&\le |\sigma^2 - \sigma_0^2| \cdot |\nabla A(\sigma_0 x)| + |\nabla A(\sigma x) - \nabla A^i(\sigma x)| \\
&\qquad + |\nabla A(\sigma_0 x) - \nabla A^i(\sigma_0 x)| + |\sigma - \sigma_0| C_i'
\end{aligned}$$

and thus

$$\|\nabla A_\sigma - \nabla A_{\sigma_0}\|_{\mathbb{1},L^p} \leq |\sigma^2 - \sigma_0^2|\sigma_0^{-\frac{n}{p}}\|\nabla A\|_{\mathbb{1},L^p} + (\sigma^{-\frac{n}{p}} + \sigma_0^{-\frac{n}{p}})\|\nabla A - \nabla A^i\|_{\mathbb{1},L^p} + |\sigma - \sigma_0|\mathrm{Vol}(B)^{\frac{1}{p}} C_i'.$$

This can again be made arbitrarily small by first choosing i and then $|\sigma - \sigma_0|$.

2.) Closedness: *Let B be equipped with a smooth metric and let $\varepsilon > 0$. Assume that a sequence $(A_i)_{i\in\mathbb{N}} \subset \mathcal{A}_\varepsilon$ converges to $A \in \mathcal{A}_\varepsilon$ and that there exist $u_i \in \mathcal{G}^{2,p}(B)$ such that $u_i^* A_i$ is in Uhlenbeck gauge with some constant C_{Uh} in (iii),(iv) for all $i \in \mathbb{N}$. Then there exists $u \in \mathcal{G}^{2,p}(U)$ such that u^*A also satisfies (i)-(iv) with the same constant C_{Uh}.*

Firstly, the L^p-energy of the connections is bounded by (A.11):

$$\|F_{A_i}\|_p \;\leq\; C\left(\|A_i\|_{W^{1,p}} + \|A_i\|_p^2\right) \;\leq\; C'$$

with a uniform constant C' due to the $W^{1,p}$-convergence of the A_i. We thus obtain a uniform $W^{1,p}$-bound on the $\tilde{A}_i := u_i^* A_i$ from (iv),

$$\|\tilde{A}_i\|_{W^{1,p}} \;\leq\; C_{Uh}\|F_{A_i}\|_p \;\leq\; C_{Uh}C'.$$

Now the Banach-Alaoglu theorem B.4 asserts that there is a subsequence (again labelled by $i \in \mathbb{N}$) of the $\tilde{A}_i$ that converges in the $W^{1,p}$-weak topology to some $\tilde{A} \in \mathcal{A}^{1,p}(B)$. Due to the compact embedding $W^{1,p} \hookrightarrow L^{2p}$ a further subsequence also converges in the L^{2p}-norm.

Next, lemma A.8 provides a uniform $W^{1,p}$-bound on $u_i^{-1}\mathrm{d}u_i = \tilde{A}_i - u_i^{-1}A_i u_i$. That lemma also asserts that we can find a further subsequence (once again labelled by $i \in \mathbb{N}$) such that u_i converges in the $\mathcal{C}^0$-topology to some $u \in \mathcal{G}^{2,p}(B)$ and at the same time $u_i^{-1}\mathrm{d}u_i$ converges in the L^{2p}-norm to $u^{-1}\mathrm{d}u$. (The latter is true since $p > \frac{n}{2}$ and hence $\frac{1}{2p} > \frac{1}{p} - \frac{1}{n}$.)

These two limits u and $\tilde{A}$ now satisfy $u^*A = \tilde{A}$ due to the uniqueness of the L^{2p}-limit

$$u^{-1}\mathrm{d}u \;\underset{\infty\leftarrow i}{\longleftarrow}\; u_i^{-1}\mathrm{d}u_i \;=\; \tilde{A}_i - u_i^{-1}A_i u_i \;\underset{i\to\infty}{\longrightarrow}\; \tilde{A} - u^{-1}Au.$$

Finally, we have to check that $\tilde{A} = u^*A$ is indeed in Uhlenbeck gauge:

(i) $\mathrm{d}^*\tilde{A}_i = 0$ holds for all $i \in \mathbb{N}$ and this equality is preserved under weak $W^{1,p}$-limits since for every $\phi \in \mathcal{C}_\delta^\infty(B)$ (which is dense in $L^p(B)$)

$$\int_B \phi * \mathrm{d}^*\tilde{A} \;=\; -\int_B \phi \cdot \mathrm{d}\big(*\tilde{A} - *\tilde{A}_i\big) \;=\; \int_B \mathrm{d}\phi \wedge *\big(\tilde{A} - \tilde{A}_i\big) \;\underset{i\to\infty}{\longrightarrow}\; 0.$$

(ii) Again $*\tilde{A}_i|_{\partial B} = 0$ holds for all $i \in \mathbb{N}$ and this is preserved under the weak $W^{1,p}$-limit. Indeed, every $\phi \in \mathcal{C}^\infty(\partial B)$ can be extended to some $\Phi \in \mathcal{C}^\infty(B)$ for which we calculate

$$\begin{aligned}\int_{\partial B} \phi \cdot *\tilde{A}|_{\partial B} &= \int_{\partial B} \Phi \cdot \big(*\tilde{A} - *\tilde{A}_i\big)\big|_{\partial B} \\ &= \int_B \mathrm{d}\Phi \wedge *\big(\tilde{A} - \tilde{A}_i\big) - \int_B \Phi \cdot \big(\mathrm{d}^*\tilde{A} - \mathrm{d}^*\tilde{A}_i\big).\end{aligned}$$

Now (i) and the convergence of $\tilde{A}_i$ to $\tilde{A}$ show that $*\tilde{A}|_{\partial B} = 0$ in the distributional sense. Here Stokes' theorem could be applied since the restriction $W^{1,p}(B) \hookrightarrow L^p(\partial B)$ is continuous (see theorem B.10) and thus a $W^{1,p}$-approximation of $\tilde{A} - \tilde{A}_i$ by smooth connections restricts to an L^p-approximation on the boundary.

(iii) The corresponding inequalities on the $\tilde{A}_i$ together with the lower semicontinuity of the norm with respect to weak limits give

$$\|\tilde{A}\|_{W^{1,q}} \;\leq\; \liminf_{i\to\infty} \|\tilde{A}_i\|_{W^{1,q}} \;\leq\; C_{Uh} \liminf_{i\to\infty} \|F_{A_i}\|_{L^q}.$$

But now the A_i converge to A with respect to the $W^{1,p}$-norm and thus also in the $W^{1,q}$-norm (since B has finite volume and $p \geq q$). Moreover, the L^q-energy $\|F_A\|_q^q$ is $W^{1,q}$-continuous (lemma A.4) and thus $\|F_{A_i}\|_{L^q}$ converges to $\|F_A\|_q$. This proves $\|\tilde{A}\|_{W^{1,q}} \leq C_{Uh}\|F_A\|_q$.

(iv) $\|\tilde{A}\|_{W^{1,p}} \leq C_{Uh}\|F_A\|_p$ follows as in (iii).

3.) Openness:
In order to prove that $\mathcal{S}_\varepsilon \subset \mathcal{A}_\varepsilon$ is open consider a connection $A \in \mathcal{A}_\varepsilon$ and a gauge transformation $u \in \mathcal{G}^{2,p}(B)$ such that u^*A is in Uhlenbeck gauge. Then the task is to find a neighbourhood of A within $\mathcal{A}_\varepsilon$, all of whose elements also can be transformed by $\mathcal{G}^{2,p}(B)$ into Uhlenbeck gauge.

If we find such a neighbourhood of u^*A within $\mathcal{A}^{1,p}(B)$ then this can be pulled back by u^* and be intersected with $\mathcal{A}_\varepsilon$ to give the required neighbourhood of A in $\mathcal{A}_\varepsilon$. This is due to the fact that $\mathcal{G}^{2,p}(B)$ acts continuously on $\mathcal{A}^{1,p}(B)$ and is closed under compositions, see the lemmata A.5 and A.6. But now u^*A has the same energy as A, so it suffices to consider connections in Uhlenbeck gauge and establish the following.

Claim: *There exist constants $\delta > 0$, $\varepsilon > 0$, and C_{Uh} such that for every metric g on B with $\|g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$ the following holds: Let $A_0 \in \mathcal{A}^{1,p}(B)$ be in Uhlenbeck gauge and have energy $\mathcal{E}(A_0) \leq \varepsilon$. Then there exists $\Delta > 0$ such that for every $A \in \mathcal{A}^{1,p}(B)$ with $\|A - A_0\|_{W^{1,p}} \leq \Delta$ a gauge transformation $u \in \mathcal{G}^{2,p}(B)$ can be found such that u^*A is in Uhlenbeck gauge.*

Note that we use the energy $\mathcal{E}$ here, not the modified energy $\mathcal{E}'$ by which $\mathcal{A}_\varepsilon$ was defined. But due to (6.1) this still suffices to prove the openness of $\mathcal{S}_{\frac{\varepsilon}{2}} \subset \mathcal{A}_{\frac{\varepsilon}{2}}$ for the choice of $\delta > 0$ involved.

Before embarking on the proof we choose $\delta > 0$ even smaller in order to obtain uniform constants in the Sobolev inequalities: The $W^{1,p}$-norm on 1-forms only depends on the metric, its inverse, and its Christoffel symbols. So one can choose $\delta > 0$ sufficiently small to ensure the following equivalence between the Euclidean $W^{1,p}$-norm on B and all other $W^{1,p}$-norms with respect to metrics g with $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$,

$$\tfrac{1}{2}\|\alpha\|_{g,W^{1,p}} \;\leq\; \|\alpha\|_{\mathbb{1},W^{1,p}} \;\leq\; 2\|\alpha\|_{g,W^{1,p}} \qquad \forall \alpha \in W^{1,p}(B, \mathrm{T}^*B). \tag{6.2}$$

Now choose $\delta > 0$ even smaller such that this equivalence holds at the same time for the $W^{1,q}$- and L^r-norm with r from lemma 6.5. Then we have uniform constants in the Sobolev estimates between these spaces (provided by the estimates for the Euclidean metric).

With these prerequisites we prove the above claim and thus the theorem: In a first step the implicit function theorem is employed to solve (i) and (ii), keeping control of some suitable Sobolev norm. The conditions (iii) and (iv) do not fit into the framework of the implicit function theorem. But in the second step one can deduce from some a priori estimate that (iii) and (iv) are satisfied automatically by our solutions of (i) and (ii).

Step 3a (Implicit function theorem): *There exists $\delta > 0$ such that for every constant C_{Uh} there exists $\varepsilon > 0$ such that for every metric g on B with $\|g - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$ the following holds:*

Let $A_0 \in \mathcal{A}^{1,p}(B)$ be a connection in Uhlenbeck gauge with $\mathcal{E}(A_0) \leq \varepsilon$, then for every $\lambda > 0$ there exists $\Delta > 0$ such that for every connection $A \in \mathcal{A}^{1,p}(B)$ with $\|A - A_0\|_{W^{1,p}} \leq \Delta$ there is a solution $V \in W^{2,p}(B, \mathfrak{g})$ of

$$\begin{cases} \mathrm{d}^*(\exp(V)^*A) &= 0 \quad \text{on } M, \\ *(\exp(V)^*A)|_{\partial B} &= 0 \quad \text{on } \partial M, \end{cases} \qquad \text{with} \qquad \|V\|_{W^{2,p}} \leq \lambda.$$

This will be proven by the implicit function theorem for the operator

$$D: \begin{array}{ccc} \mathcal{A}^{1,p}(B) \times W^{2,p}_m(B, \mathfrak{g}) & \longrightarrow & \mathcal{Z} \\ (A, V) & \longmapsto & \big(\mathrm{d}^*(\exp(V)^*A)\,,\, *(\exp(V)^*A)|_{\partial B}\big) \end{array}$$

on the Banach spaces

$$W^{2,p}_m(B, \mathfrak{g}) := \big\{ V \in W^{2,p}(B, \mathfrak{g}) \;\big|\; \textstyle\int_B V = 0 \big\},$$
$$\mathcal{Z} := \big\{ (f, \phi) \in L^p(B, \mathfrak{g}) \times W^{1,p}_\partial(B, \mathfrak{g}) \;\big|\; \textstyle\int_B f + \int_{\partial B} \phi = 0 \big\}.$$

Of course this requires some explanations: Firstly, $\mathcal{A}^{1,p}(B)$ is a Banach space as are all Sobolev spaces, and so is $W^{2,p}_m(B, \mathfrak{g})$ as a closed subset of $W^{2,p}(B, \mathfrak{g})$. The space $W^{1,p}_\partial(B,\mathfrak{g})$ is defined, as in chapter 3 for real values, to be the quotient of $W^{1,p}(B,\mathfrak{g})$ by its subspace of functions vanishing at the boundary. It is a Banach space when equipped with the quotient norm

$$\|\phi\|_{W^{1,p}_\partial} = \inf\bigl\{\|\Phi\|_{W^{1,p}} \bigm| \Phi \in W^{1,p}(B,\mathfrak{g}), \Phi|_{\partial B} = \phi\bigr\}.$$

Moreover, theorem B.10 asserts that the embedding $W^{1,p}_\partial(B,\mathfrak{g}) \hookrightarrow L^1(\partial B,\mathfrak{g})$ is continuous, hence the equation $\int_B f = -\int_{\partial B}\phi$ for $(f,\phi)\in\mathcal{Z}$ is preserved under limits in $L^p(B,\mathfrak{g})\times W^{1,p}_\partial(B,\mathfrak{g})$. So $\mathcal{Z}$ also is a Banach space as a closed subset of the Cartesian product of two Banach spaces.

The second component of D is understood as follows: Let ν be the outward unit normal vector field to ∂B. Then one can use lemma 5.6 (i) to identify $*(\exp(V)^*A)|_{\partial B} = \phi\,\mathrm{dvol}_{\partial B}$ with the function $\phi = (\exp(V)^*A)(\nu)$ on ∂B. This indeed defines a function in the space $W^{1,p}_\partial(M,\mathfrak{g})$ since $\phi = \Phi|_{\partial B}$ with $\Phi = (\exp(V)^*A)(\tilde\nu)\in W^{1,p}(M,\mathfrak{g})$ for any smooth extension $\tilde\nu$ of ν to B.

Now consider $V\in W^{2,p}(B,\mathfrak{g})$, then by definition $\exp(V)\in\mathcal{G}^{2,p}(B)$ and by lemma A.6 this gauge transformation acts continuously on $\mathcal{A}^{1,p}(B)$. Lemma A.6 moreover asserts that this gauge action depends continuously on $\exp(V)\in\mathcal{G}^{2,p}(B)$, and this in turn depends continuously on $V\in W^{2,p}(B,\mathfrak{g})$ by definition (iii) in lemma B.7. Thus D maps continuously to $L^p(B,\mathfrak{g})\times W^{1,p}_\partial(B,\mathfrak{g})$. Furthermore, Stokes' theorem yields the equality that defines $\mathcal{Z}$:

$$\int_B \mathrm{d}^*A_i\,\mathrm{d}vol_B \;=\; -\int_B \mathrm{d}*A_i \;=\; -\int_{\partial B} *A_i \;=\; -\int_{\partial B} A_i(\nu)\,\mathrm{d}vol_{\partial B}.$$

Here Stokes' theorem extends to the nonsmooth A_i since it is a $W^{1,p}$-limit of smooth 1-forms; and due to the continuous embedding $W^{1,p}(B)\hookrightarrow L^1(\partial B)$ from theorem B.10 this also represents $A_i(\nu)$ as L^1-limit of smooth functions on ∂B. Thus D actually maps into $\mathcal{Z}$, so it is a well defined continuous map between Banach spaces.

We want to apply the implicit function theorem E.1 to find the zeros of D close to $D(A_0,0)=(0,0)$. Thus we have to examine the partial derivative of D with respect to the second argument,

$$\partial_2 D_{(A,V)} : \begin{array}{ccc} W^{2,p}_m(B,\mathfrak{g}) & \to & \mathcal{Z} \\ \xi & \mapsto & \bigl(\mathrm{d}^*\mathcal{G}(A,V,\xi)\,,\, *\mathcal{G}(A,V,\xi)\big|_{\partial B}\bigr) \end{array}$$

with the linearization of the gauge action

$$\mathcal{G}(A,V,\xi) \;=\; \mathrm{d}\xi + \mathrm{d}_{\exp(-V)}\mathrm{Ad}\,\bigl(\mathrm{d}_{-V}\exp(-\xi)\bigr)\;A \;\in \mathcal{A}^{1,p}(B,\mathfrak{g}).$$

One can check that $\partial_2 D_{(A,V)}$ depends continuously on both $A\in\mathcal{A}^{1,p}(B)$ and on $V\in W^{2,p}(B,\mathfrak{g})$. The main task is then to show that at $(A_0,0)$ this map is bijective. Fortunately the expression simplifies considerably at this point:

$$\begin{aligned}\mathcal{G}(A_0,0,\xi) &= \mathrm{d}\xi + \mathrm{d}_{\mathbb{1}}\mathrm{Ad}\,\bigl(\mathrm{d}_0\exp(-\xi)\bigr)\;A_0\\ &= \mathrm{d}\xi + \mathrm{ad}_{-\xi}(A_0)\\ &= \mathrm{d}\xi - [\xi\wedge A_0].\end{aligned}$$

Here $[\xi\wedge A_0]$ denotes the wedge product of the two forms with the Lie bracket used to combine the values in $\mathfrak{g}$ as in appendix A. Note that $*[\xi\wedge A_0]\big|_{\partial B} = [\xi\wedge *A_0]\big|_{\partial B}$ vanishes since A_0 is in Uhlenbeck gauge, and moreover

$$-\mathrm{d}^*[\xi\wedge A_0] \;=\; *[\mathrm{d}\xi\wedge *A_0] - [\xi\wedge \mathrm{d}^* A_0] \;=\; *[\mathrm{d}\xi\wedge *A_0].$$

So $\partial_2 D_{(A_0,0)}$ is the following map:

$$\begin{array}{ccc} W^{2,p}_m(B,\mathfrak{g}) & \to & \mathcal{Z} \\ \xi & \mapsto & \big(\,\mathrm{d}^*\mathrm{d}\xi + *[\mathrm{d}\xi\wedge *A_0]\;,\; *\mathrm{d}\xi\big|_{\partial B}\,\big)\;. \end{array}$$

In order to prove the bijectivity of this operator we consider it as perturbation $\partial_2 D_{(A_0,0)} = T + S$ with

$$T:\xi\mapsto \big(\Delta\xi\,,\,\tfrac{\partial\xi}{\partial\nu}\big), \qquad\qquad S:\xi\mapsto\big(*[\mathrm{d}\xi\wedge *A_0]\,,\,0\big).$$

Here $\Delta = \mathrm{d}^*\mathrm{d}$ is the Hodge Laplace operator on functions with respect to the metric g. Now T is the operator of the inhomogeneous Neumann problem (with values in $\mathfrak{g}$), so by theorem 3.1 it is a surjection from $W^{2,p}(B,\mathfrak{g})$ onto $\mathcal{Z}$ – the equation that defines $\mathcal{Z}$ is the necessary and sufficient condition (3.2) for the solvability of the inhomogeneous Neumann problem (3.1). Corollary 1.9 then says that these solutions are unique up to an additive constant, hence T becomes a bijection by restriction to $W^{2,p}_m(B,\mathfrak{g})$. Finally, the meaning of the Agmon-Douglis-Nirenberg estimate in theorem 3.2 is just that T^{-1} is bounded. In fact, we have proven that there is a uniform bound on the operators T^{-1} for all metrics with sufficiently small $\|g-\mathbb{1}\|_{W^{2,\infty}}$. So we choose $\delta>0$ appropriately to obtain $\|T^{-1}\|\le C_T$ with some constant C_T that is independent of the metric. [1] Moreover, since T maps into $\mathcal{Z}$ the perturbation S also is a linear operator from $W^{1,p}(B,\mathfrak{g})$ to $\mathcal{Z}$. So it remains to find a bound on S such that lemma E.4 yields the bijectivity of $T+S$: We will prove

$$\|S\xi\|_{\mathcal{Z}} \le \tfrac{1}{2C_T}\|\xi\|_{W^{2,p}} \qquad \forall\xi\in W^{2,p}(B,\mathfrak{g}). \tag{6.3}$$

Let A_0 be given by $A_i\mathrm{d}x^i$ in the standard coordinates of $B\subset\mathbb{R}^n$, then for all $\xi\in W^{2,p}(B,\mathfrak{g})$ we can use a property (A.5) of the norm on $\mathfrak{g}$ to obtain

$$\big|*[\mathrm{d}\xi\wedge *A_0]\big| \;=\; \big|g^{ij}[\partial_i\xi,A_j]\big| \;\le\; |g^{-1}|\cdot|\mathrm{d}\xi|\cdot|A_0|.$$

Now require $\delta\le\frac12$ such that $|g^{-1}|\le(1-\|g-\mathbb{1}\|_\infty)^{-1}\le 2$ on all of B. But note that here the pointwise norm on the 1-forms $\mathrm{d}\xi$ and A is with respect to

[1] This is the only point at which control of the second derivatives of the metric is needed. In fact, one can get around this by using the Euclidean Neumann operator as unperturbed operator T. In that case the Banach spaces in the implicit function theorem also have to be defined in terms of the Euclidean metric and the difference between the Euclidean and the actual Neumann operator enters in the perturbed operator S.

the Euclidean metric on B. So we have to apply the Euclidean L^p-norm to this equality and use the equivalence of the metrics (6.2): For all $\xi \in W^{2,p}(B,\mathfrak{g})$

$$\begin{aligned} \|S\xi\|_{\mathcal{Z}} &= \big\|{*}[\mathrm{d}\xi \wedge {*}A_0]\big\|_p \\ &\leq 2\big\|{*}[\mathrm{d}\xi \wedge {*}A_0]\big\|_{\mathbb{1},L^p} \\ &\leq 4\big\||\mathrm{d}\xi| \cdot |A_0|\big\|_{\mathbb{1},L^p} \\ &\leq 4C_S\big\||A_0|\big\|_{\mathbb{1},W^{1,q}}\big\||\mathrm{d}\xi|\big\|_{\mathbb{1},W^{1,p}} \\ &\leq 16C_S\|A_0\|_{W^{1,q}}\|\xi\|_{W^{2,p}}. \end{aligned}$$

Here we used lemma 6.5 and $\big|\nabla|A|\big| \leq |\nabla A|$, which holds for the Euclidean metric on B.

Finally, let C_{Uh} be given, then choose $\varepsilon = (32\,C_S C_{Uh} C_T)^{-q}$ and use the Uhlenbeck gauge condition (iii) $\|A_0\|_{W^{1,q}} \leq C_{Uh}\|F_{A_0}\|_q$ and energy bound $\mathcal{E}(A_0) \leq \varepsilon$ to establish (6.3):

$$\|S\xi\|_{\mathcal{Z}} \;\leq\; 16\,C_S C_{Uh}\varepsilon^{\frac{1}{q}}\|\xi\|_{W^{2,p}} \;=\; \tfrac{1}{2C_T}\|\xi\|_{W^{2,p}} \qquad \forall \xi \in W^{2,p}(B,\mathfrak{g}).$$

Now lemma E.4 asserts that $\partial_2 D_{(A_0,0)}$ is bijective. So all conditions of the implicit function theorem E.1 are satisfied by the operator D at $(A_0,0)$, where $D(A_0,0) = (0,0)$, and this finally proves the claim:

We find a neighbourhood $\mathcal{U} \subset \mathcal{A}^{1,p}(B)$ of A_0 such that for every $A \in \mathcal{U}$ there exists a solution $V \in W^{2,p}(B,\mathfrak{g})$ of $D(A,V) = (0,0)$, that is $\exp(V)^*A$ meets the Uhlenbeck gauge conditions (i) and (ii). Furthermore, this solution is given by a continuous map to a neighbourhood of $0 \in W^{2,p}_m(B,\mathfrak{g})$. Hence for every given $\lambda > 0$ there exists a ball of sufficiently small radius $\Delta > 0$ in $\mathcal{U}$ such that for all A within this ball the corresponding solution V meets $\|V\|_{W^{2,p}} \leq \lambda$.

Step 3b (A priori estimate): *There exist constants $\delta > 0$, C_{Uh}, and $\kappa > 0$ such that for every metric g on B with $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$ the following holds: Let $A \in \mathcal{A}^{1,p}(B)$ solve (i) and (ii), i.e.*

$$\begin{cases} \mathrm{d}^*A = 0 & \text{on } M, \\ {*}A|_{\partial B} = 0 & \text{on } \partial M, \end{cases} \qquad \text{with} \qquad \|A\|_r \leq \kappa,$$

where $r = r(n,q,p)$ from lemma 6.5. Then A also solves (iii) and (iv):

$$\|A\|_{W^{1,p}} \leq C_{Uh}\|F_A\|_p, \qquad \|A\|_{W^{1,q}} \leq C_{Uh}\|F_A\|_q.$$

The key to this is theorem 5.1. It certainly generalizes to 1-forms with values in the finite dimensional Lie algebra $\mathfrak{g}$. Furthermore, the constants depend $W^{1,\infty}$-continuously on the metric, hence we can choose $\delta > 0$ such that the estimate holds with a uniform constant C_e for all metrics g with $\|g - \mathbb{1}\|_{W^{1,\infty}} \leq \delta$ and for both Sobolev exponents $s = p$ and $s = q$:

$$\|A\|_{W^{1,s}} \leq C_e\left(\|\mathrm{d}A\|_s + \|\mathrm{d}^*A\|_s\right)$$

for all connections $A \in \mathcal{A}^{1,p}(B)$ with $*A|_{\partial B} = 0$. Now assume $A \in \mathcal{A}^{1,p}(B)$ to solve (i) and (ii), then this estimate applies and we can bring the curvature $F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A]$ into play:

$$\|A\|_{W^{1,s}} \;\leq\; C_e \|\mathrm{d}A\|_s \;\leq\; C_e \left(\|F_A\|_s + \tfrac{1}{2}\big\|[A \wedge A]\big\|_s\right).$$

The second term is bounded by $\big\||A| \cdot |A|\big\|_s$ as seen in lemma A.4, so we can apply lemma 6.5. But in order to obtain metric independent constants we use the equivalence of the norms (6.2) and use the lemma only for the Euclidean metric: With $r = r(n, q, p)$ from lemma 6.5

$$\begin{aligned} \tfrac{1}{2}\big\|[A \wedge A]\big\|_s &\leq 2\big\||A|_{\mathbb{1}} \cdot |A|_{\mathbb{1}}\big\|_{\mathbb{1},L^s} \\ &\leq 2C_S \|A\|_{\mathbb{1},L^r} \|A\|_{\mathbb{1},W^{1,s}} \\ &\leq 8C_S \|A\|_r \|A\|_{W^{1,s}}. \end{aligned}$$

As before we also used $\big|\nabla|A|\big| \leq |\nabla A|$ for the Euclidean metric. Thus for some finite, metric independent constant C

$$\|A\|_{W^{1,s}} \leq C\big(\|F_A\|_s + \|A\|_r \|A\|_{W^{1,s}}\big).$$

Now choose $\kappa = \frac{1}{2}C^{-1}$, then the finite term $C\|A\|_r\|A\|_{W^{1,s}} \leq \frac{1}{2}\|A\|_{W^{1,s}}$ can be absorbed into the left hand side and the result follows with $C_{Uh} = 2C$.

In order to prove the claim (and thus establish the openness and the whole theorem) the result from step 3b has to be applied to the solution $\exp(V)^*A$ from step 3a. This solution certainly satisfies (i) and (ii) and

$$\|\exp(V)^*A\|_r \leq \|\mathrm{Ad}_{\exp(V)}A\|_r + \|\exp(-V)\mathrm{d}\exp(V)\|_r. \tag{6.4}$$

If moreover this is bounded by κ then (iii) and (iv) follow from step 3b, hence $\exp(V)^*A$ is in Uhlenbeck gauge. In order to achieve this bound we have to choose an appropriate $\lambda > 0$ (that is allowed to depend on the metric) in step 3a, and we can choose even smaller uniform constants $\varepsilon > 0$ and $\Delta > 0$. The first term in (6.4) is just $\|A\|_r$ due to the invariance (A.4) of the metric on $\mathfrak{g}$, and this is estimated by

$$\begin{aligned} \|A\|_r &\leq \|A - A_0\|_r + \|A_0\|_r \\ &\leq C\big(\|A - A_0\|_{W^{1,p}} + \|A_0\|_{W^{1,q}}\big) \\ &\leq C(\Delta + C_{Uh}\varepsilon). \end{aligned}$$

Here we used the Sobolev inequalities for $W^{1,q} \hookrightarrow L^r$ and $W^{1,p} \hookrightarrow L^r$ with a uniform constant C due to the equivalence of the norms (6.2). So the first term can be made as small as $\frac{\kappa}{2}$ by the choice of $\varepsilon > 0$ and $\Delta > 0$ (for the choice of ε note that κ also is a uniform constant). For the second term in (6.4) note that $\lambda > 0$ bounds the $\mathcal{C}^0$-norm of V due to the Sobolev embedding $W^{2,p} \hookrightarrow \mathcal{C}^0$. So if $\lambda > 0$

is sufficiently small then for every $x \in B$ the map $\exp(-V(x))\mathrm{d}_{V(x)}\exp : \mathfrak{g} \to \mathfrak{g}$ is arbitrarily close to $\mathrm{d}_0 \exp = \mathrm{Id}_{\mathfrak{g}}$, thus its norm can be bounded by 2. Hence again with a Sobolev embedding $W^{1,p} \hookrightarrow L^r$

$$\begin{aligned}\|\exp(-V)\mathrm{d}\exp(V)\|_r &= \|\exp(-V)\mathrm{d}_V \exp\circ \mathrm{d}V\|_r \\ &\leq 2\|\mathrm{d}V\|_r \\ &\leq 2C\|V\|_{W^{2,p}} \;\leq\; 2C\lambda.\end{aligned}$$

So for sufficiently small $\lambda > 0$ in step 3a the solution $u = \exp(V) \in \mathcal{G}^{2,p}(B)$ from the implicit function theorem also meets the Uhlenbeck gauge conditions (iii) and (iv) which finally proves claim for the openness and thus the theorem. □

Now it remains to examine the effect of the rescaling of the metric in order to prove the Uhlenbeck gauge theorem.

Proof of theorem 6.1 :
Let g_M be the metric on M and let a point in M be given. Then as explained in the beginning of this chapter there exist local coordinates ψ_σ from the ball or the "egg" $B \subset \mathbb{R}^n$ to a neighbourhood $U = \psi_\sigma(B)$ of this point such that $\|\sigma^{-2}\psi_\sigma^* g_M - \mathbb{1}\|_{W^{2,\infty}} \leq \delta$. Now $\delta > 0$ can be chosen sufficiently small for theorem 6.3 to hold. Then the theorem holds on B with the metric $g = \sigma^{-2}\psi_\sigma^* g_M$, but the real metric, with respect to which we wish to prove the existence of Uhlenbeck gauge, is $\sigma^2 g$ for some $\sigma \in (0, 1]$. Note that it suffices to prove the theorem in some coordinate chart since all conditions and properties are invariant under the change of coordinates.

So we have to consider the effect of this rescaling on the energy and the inequalities (iii),(iv). The spaces $\mathcal{A}^{1,p}(B), \mathcal{G}^{2,p}(B)$, and the equations (i),(ii) are invariant under constant conformal change of the metric. For the energy (with respect to the metric indicated by the subscript) of a connection $A \in \mathcal{A}^{1,p}(B)$

$$\mathcal{E}_{\sigma^2 g}(A) \;=\; \int_B \left(\sigma^{-2}g^{ik}\sigma^{-2}g^{jl}(F_A)_{ij}(F_A)_{kl}\right)^{\frac{q}{2}} \sqrt{\det(\sigma^2 g)}\,\mathrm{d}^n x \;=\; \sigma^{n-2q}\mathcal{E}_g(A).$$

So if $\mathcal{E}_{\sigma^2 g}(A) \leq \varepsilon_{Uh}$ then due to $q \geq \frac{n}{2}$ and $\sigma \leq 1$ we also have $\mathcal{E}_g(A) \leq \varepsilon_{Uh}$ and hence the Uhlenbeck gauge theorem in the model case applies and gives a gauge transformation that transforms A into Uhlenbeck gauge with respect to the metric g.

Moreover, if a connection $A \in \mathcal{A}^{1,p}(B)$ satisfies $\|A\|_{g,W^{1,p}} \leq C_{Uh}\|F_A\|_{g,L^p}$ for $p \geq \frac{n}{2}$ (i.e. it satisfies (iii) or (iv) with respect to the metric g) then it also satisfies the same inequality with respect to $\sigma^2 g$. Indeed,

$$\|A\|^p_{\sigma^2 g, L^p} \;=\; \int_B \left(\sigma^{-2}g^{ij}A_i A_j\right)^{\frac{p}{2}} \sqrt{\det(\sigma^2 g)}\,\mathrm{d}^n x \;=\; \sigma^{n-p}\|A\|^p_{g,L^p}.$$

Furthermore, note that g and $\sigma^2 g$ have the same Christoffel symbols Γ^i_{jk}, so

$$\begin{aligned}
&\|\nabla^{\sigma^2 g} A\|^p_{\sigma^2 g, L^p} \\
&= \int_B \left(\sigma^{-2} g^{ik} \sigma^{-2} g^{jl} \left(\tfrac{\partial A_j}{\partial x^i} + \Gamma^m_{ij} A_m\right)\left(\tfrac{\partial A_l}{\partial x^k} + \Gamma^n_{kl} A_n\right)\right)^{\frac{p}{2}} \sqrt{\det(\sigma^2 g)}\, \mathrm{d}^n x \\
&= \sigma^{n-2p} \|\nabla^g A\|^p_{g,L^p}.
\end{aligned}$$

Thus $\|A\|_{\sigma^2 g, W^{1,p}} \leq \sigma^{\frac{n-2p}{p}} \|A\|_{g,W^{1,p}}$ since $\sigma \leq 1$ and for the L^p-norm of the curvature we obtain as for the energy $\|F_A\|_{\sigma^2 g, L^p} = \sigma^{\frac{n-2p}{p}} \|F_A\|_{g,L^p}$, hence the inequalities (iii) and (iv) carry over to the metric $\sigma^2 g$ with the same constant C_{Uh}.
□

Before establishing the extension of theorem 6.1 in remark 6.2 a) we first sketch the definition of $W^{2,\frac{n}{2}}(M,G)$. For this purpose we fix an embedding $G \hookrightarrow \mathbb{R}^m$. It is given by the Whitney theorem (e.g. [H, Thm.2.2.14]), and since G is a compact Lie group we can think of $G \subset \mathbb{R}^{m\times m}$ as a matrix group. The standard example is $G = \mathrm{SU}(2) \subset \mathbb{C}^{2\times 2}$. Now $W^{2,\frac{n}{2}}(M,G)$ can be defined as the compactification of $\mathcal{C}^\infty(M,G)$ with respect to the $W^{2,\frac{n}{2}}$-norm on $\mathcal{C}^\infty(M,\mathbb{R}^m)$. However, this definition is not independent of the choice of the embedding of G. This is only true for Sobolev spaces that embed into the continuous maps, cf. appendix B.

In the standard case $G = \mathrm{SU}(2)$, the limit $u \in W^{2,\frac{n}{2}}(M,\mathbb{C}^{2\times 2})$ of a $W^{2,\frac{n}{2}}$-Cauchy series $u_i \in \mathcal{C}^\infty(M,\mathrm{SU}(2))$ will satisfy $u^* u = \mathbb{1}$ in the sense of L^p for any $p < \infty$. This is since u_i and u_i^* will converge in the L^{2p}-norm. That also means that $u^{-1} = u^* \in W^{2,\frac{n}{2}}(M,\mathbb{C}^{2\times 2})$ is welldefined.

Proof of remark 6.2 a) :
We set $q = \frac{n}{2}$, and choose $p = n$. Then theorem 6.1 provides constants C_{Uh} and $\varepsilon_{Uh} > 0$ and a neighbourhood U of any fixed point such that every connection $A \in \mathcal{A}^{1,n}(U)$ with $\mathcal{E}(A) \leq \varepsilon_{Uh}$ can be put into Uhlenbeck gauge.

Now consider $A \in \mathcal{A}^{1,\frac{n}{2}}(U)$ with $\mathcal{E}(A) \leq \frac{1}{2}\varepsilon_{Uh}$. The claim is that there exists a gauge transformation $u \in \mathcal{G}^{2,\frac{n}{2}}(U)$ such that $u^* A$ satisfies

$$\begin{cases} \mathrm{d}^*(u^* A) = 0, \\ *(u^* A)|_{\partial U} = 0, \end{cases} \qquad \text{and} \qquad \|u^* A\|_{W^{1,\frac{n}{2}}} \leq C_{Uh} \|F_A\|_{\frac{n}{2}}.$$

To prove this we choose a sequence of smooth connections $A_i \in \mathcal{A}(U)$ that converge to A in the $W^{2,\frac{n}{2}}$-norm. For sufficiently large i these will satisfy $\mathcal{E}(A_i) \leq \varepsilon_{Uh}$ since $F_{A_i} = \mathrm{d}A_i + \frac{1}{2}[A_i \wedge A_i]$ converges to F_A in the $L^{\frac{n}{2}}$-norm. This is due to the Sobolev embedding $W^{1,\frac{n}{2}}(M) \hookrightarrow L^n(M)$ that will be used throughout this proof without further mentioning.

Now theorem 6.1 provides gauge transformations $u_i \in \mathcal{G}^{2,n}(U)$ that put $u_i^* A_i$ into Uhlenbeck gauge. The gauge action can be rewritten as

$$u_i^{-1} \mathrm{d}u_i = u_i^* A_i - u_i^{-1} A_i u_i.$$

From this one obtains a uniform bound

$$\|\mathrm{d}u_i\|_n \;=\; \|u_i^{-1}\mathrm{d}u_i\|_n \;\le\; C\|u_i^*A_i\|_{W^{1,\frac{n}{2}}} + \|u_i^{-1}A_iu_i\|_n$$
$$\le\; C\,C_{Uh}\|F_{A_i}\|_{\frac{n}{2}} + \|A_i\|_n \;\le\; C'.$$

This uses the fact that F_{A_i} and A_i converge in the $L^{\frac{n}{2}}$-norm and the L^n-norm respectively. One also finds a uniform bound

$$\begin{aligned}\|\nabla^2 u_i\|_{\frac{n}{2}} &= \|\nabla(u_i^{-1}\mathrm{d}u_i)\|_{\frac{n}{2}} + \|\nabla u_i \otimes u_i^{-1}\mathrm{d}u_i\|_{\frac{n}{2}} \\ &\le\; \|u_i^*A_i\|_{W^{1,\frac{n}{2}}} + \|u_i^{-1}A_iu_i\|_{W^{1,\frac{n}{2}}} + \|\mathrm{d}u_i\|_n^2 \\ &\le\; C_{Uh}\|F_{A_i}\|_{\frac{n}{2}} + \|A_i\|_{\frac{n}{2}} + \|\nabla A_i\|_{\frac{n}{2}} + 2\|A_i\|_n\|\mathrm{d}u_i\|_n + \|\mathrm{d}u_i\|_n^2 \;\le\; C'.\end{aligned}$$

Since the u_i moreover take values in a compact Lie group this implies that (after going to a subsequence) one has a weak $W^{2,\frac{n}{2}}$-limit $u \in \mathcal{G}^{2,\frac{n}{2}}$. The convergence $u_i \to u$ also holds in the weak $W^{1,n}$-topology and in the L^p-norm for any $p < \infty$ due to the compact Sobolev embedding $W^{2,\frac{n}{2}} \hookrightarrow L^p$. (The weak convergence is deduced from the standard Banach-Alaoglu theorem B.4.) So the $u_i^*A_i = u_i^{-1}A_iu_i + u_i^{-1}\mathrm{d}u_i$ will converge $W^{1,s}$-weakly to u^*A for any $1 \le s < \frac{n}{2}$.

Now we can check the Uhlenbeck gauge conditions for u^*A as in the 'closedness' part of the proof of theorem 6.1: $\mathrm{d}^*(u^*A) = 0$ holds in the sense of $L^s(M)$ and $*(u^*A)|_{\partial U} = 0$ holds in the sense of $L^s(\partial M)$. The latter is due to the embedding $W^{1,s}(M) \hookrightarrow L^s(\partial M)$ provided by theorem B.10). The estimate also follows from the estimate on the $u_i^*A_i$: For all $i \in \mathbb{N}$

$$\|u_i^*A_i\|_{W^{1,\frac{n}{2}}} \;\le\; C_{Uh}\|F_{u_i^*A_i}\|_{\frac{n}{2}} \;=\; C_{Uh}\|F_{A_i}\|_{\frac{n}{2}}.$$

This is uniformly bounded and in fact converges to $C_{Uh}\|F_A\|_{\frac{n}{2}}$ since $F_{A_i} \to F_A$ in the $L^{\frac{n}{2}}$-norm. So by theorem B.4 there must be a weakly $W^{1,\frac{n}{2}}$-convergent subsequence of the $u_i^*A_i$. Its limit can only be u^*A since that already was the weak $W^{1,s}$-limit. Now the lower semicontinuity of the norm with respect to weak limits implies the claimed $\|u^*A\|_{W^{1,\frac{n}{2}}} \le C_{Uh}\|F_A\|_{\frac{n}{2}}$. □

Chapter 7

Patching

In this chapter we finally prove the weak Uhlenbeck compactness theorem A and its generalization, theorem A′, to noncompact manifolds. This requires two different types of patching constructions for gauge transformations:

For theorem A (on compact base manifolds), one has to patch together the local Uhlenbeck gauges provided by theorem B (proven as theorem 6.1). For theorem A′ (on noncompact base manifolds), one moreover has to patch together gauge transformations provided by theorem A on compact submanifolds. In both cases, one has to extend given gauge transformations to larger domains. This cannot simply be achieved by cutoff functions since the gauge transformations take values in a Lie group.

One way of dealing with this difficulty is to make sure that the gauge transformation lies within a coordinate chart of the Lie group, in which one can use a cutoff function. This will be the case for the first patching construction in lemma 7.2. This construction is used to prove theorem A, and it can also be used for a proof of the strong Uhlenbeck compactness theorem E on compact manifolds.

Another topological way of extending gauge transformations is to choose the domains such that the extension is always possible. This is the case in proposition 7.6, where the domains are deformation retracts of the manifold. This patching construction will be used for both the generalization of the weak and strong Uhlenbeck compactness to noncompact manifolds in theorem A′ and theorem E′.

The general setting in this chapter for the weak Uhlenbeck compactness is the following: Let G be a compact Lie group. Then consider a principal G-bundle $P \to M$, where the base manifold M is compact in the case of theorem A and noncompact (but exhausted by compact sets) in theorem A′.

First assume that the base manifold M is a compact Riemannian n-manifold. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$. Denote by $\mathcal{A}^{1,p}(P)$ the $W^{1,p}$-Sobolev space of connections and by $\mathcal{G}^{2,p}(P)$ the $W^{2,p}$-Sobolev space of gauge transformations on P. These and the action of $\mathcal{G}^{2,p}(P)$ on $\mathcal{A}^{1,p}(P)$ are well defined as in appendix B. Now we restate and prove the weak Uhlenbeck compactness theorem A.

Theorem 7.1 (Weak Uhlenbeck Compactness)
Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P)$ be a sequence of connections such that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence $u^\nu \in \mathcal{G}^{2,p}(P)$ of gauge transformations such that $u^{\nu\,*}A^\nu$ weakly converges in $\mathcal{A}^{1,p}(P)$.

We first prove this theorem up to the actual patching argument that is postponed to the subsequent lemma 7.2. The purpose of this proceeding is to naturally develop the setting for the patching before we go into the technicalities of the proof.

Proof of theorem 7.1 :
Choose $1 < q < p$ such that $q \geq \frac{n}{2}$ and $q \geq \frac{pn}{p+n}$. This is possible since $p > \frac{n}{2}$ and $p > \frac{pn}{p+n}$ and it ensures that the local gauge theorem 6.1 holds on M with the L^q-energy $\mathcal{E}$. Let C_{Uh} and ε_{Uh} be the constants from that theorem and consider the energy of the connections A^ν over some small trivializing chart $U \subset M$:

$$\mathcal{E}(A^\nu|_U) = \int_U |F_{A^\nu}|^q \leq (\mathrm{Vol}\,U)^{1-\frac{q}{p}} \|F_{A_\nu}\|_p^q$$

This is less than ε_{Uh} whenever U has sufficiently small volume – independently of $\nu \in \mathbb{N}$ due to the uniform bound on $\|F_{A_\nu}\|_p$. Now for every point in M fix a neighbourhood of such small volume over which the bundle P is trivial. Then the local gauge theorem 6.1 with remark 6.2 e) asserts that for every point in M there exists a trivialization over an even smaller neighbourhood such that in addition the assertion of the theorem is true (i.e. all connections with sufficiently small energy can be put into Uhlenbeck gauge). Since M is compact it is covered by finitely many of these neighbourhoods, $M = \bigcup_{\alpha=1}^N U_\alpha$.

The trivializations over the U_α form a bundle atlas of P. With respect to this atlas the connections A^ν are represented by 1-forms $A^\nu_\alpha \in \mathcal{A}^{1,p}(U_\alpha)$ with $\mathcal{E}(A^\nu_\alpha) = \mathcal{E}(A^\nu|_{U_\alpha}) \leq \varepsilon_{Uh}$ due to the small volume of the U_α. Thus all A^ν_α can be put into Uhlenbeck gauge, that is for all $\alpha = 1, \dots, N$ and $\nu \in \mathbb{N}$ there exists a local gauge transformation $u^\nu_\alpha \in \mathcal{G}^{2,p}(U_\alpha)$ such that $u^{\nu\,*}_\alpha A^\nu_\alpha$ satisfies the Uhlenbeck gauge conditions. In particular, $\|u^{\nu\,*}_\alpha A^\nu_\alpha\|_{W^{1,p}} \leq C_{Uh}\|F_{A^\nu_\alpha}\|_p$, and this is uniformly bounded for all $\nu \in \mathbb{N}$ due to the uniform bound on $\|F_{A^\nu}\|_{L^p(M)} \geq \|F_{A^\nu_\alpha}\|_{L^p(U_\alpha)}$. Hence on all U_α there exist weakly convergent subsequences of the $u^{\nu\,*}_\alpha A^\nu_\alpha$. However, the u^ν_α do not define global gauge transformations, i.e. bundle isomorphisms. This would only be the case if on all intersections $U_\alpha \cap U_\beta$

$$u^\nu_{\alpha\beta} := (u^\nu_\alpha)^{-1}\phi_{\alpha\beta}u^\nu_\beta$$

were identical to the transition function $\phi_{\alpha\beta}$ of the bundle atlas. If the action of u^ν_α is viewed as a change of the trivialization over U_α then the $u^\nu_{\alpha\beta}$ are the new transition functions and thus satisfy the cocycle conditions (A.1).

In general the $u^\nu_{\alpha\beta}$ can not even be expected to be $\mathcal{C}^0$-close to the $\phi_{\alpha\beta}$. The Uhlenbeck gauge conditions only fix the u^ν_α up to a constant gauge transformation;

so at a fixed point every value of $u^\nu_{\alpha\beta}$ can be achieved by the choice of these constants. However, in order to use cutoff functions for the patching of the u^ν_α to a global gauge transformation we have to be able to work in the Lie algebra, i.e. (via the exponential map) in geodesic balls of the Lie group. So that way we can only achieve $\mathcal{C}^0$-small corrections of the $u^\nu_{\alpha\beta}$. The possible constant changes etc. described above have to be compensated separately from the actual patching. Now the key observation is that on all intersections $U_\alpha \cap U_\beta$

$$u^\nu_{\alpha\beta}{}^*\left(u^\nu_\alpha{}^* A^\nu_\alpha\right) = u^\nu_\beta{}^* A^\nu_\beta .$$

This follows from the transition identity $A^\nu_\beta = \phi_{\alpha\beta}{}^* A^\nu_\alpha$ for the representatives of the connections A^ν. Since both $u^\nu_\alpha{}^* A^\nu_\alpha$ and $u^\nu_\beta{}^* A^\nu_\beta$ are uniformly bounded in the $W^{1,p}$-norm on $U_\alpha \cap U_\beta$ (even on U_α and U_β respectively) lemma A.8 applies (to the trivial bundle $P|_{U_\alpha\cap U_\beta}$). It asserts that there are uniform bounds on $\|(u^\nu_{\alpha\beta})^{-1}\mathrm{d}u^\nu_{\alpha\beta}\|_{W^{1,p}}$ and that some subsequence of the $u^\nu_{\alpha\beta}$ converges $\mathcal{C}^0$-uniformly on all (finitely many) intersections $U_\alpha \cap U_\beta$. Thus for every $\delta > 0$ there exists a subsequence (again labelled by $\nu \in \mathbb{N}$) such that the transition functions all lie within a geodesic δ-ball of one another: Denote $g_\alpha := u^1_\alpha$ with the corresponding transition functions $g_{\alpha\beta} = g_\alpha^{-1}\phi_{\alpha\beta}g_\beta = u^1_{\alpha\beta}$, then for all $\alpha,\beta = 1,\dots,N$ and $\nu \in \mathbb{N}$ we have

$$\mathrm{d}(u^\nu_{\alpha\beta}\,,\, g_{\alpha\beta}) \le \delta .$$

Here d denotes the supremum over $U_\alpha \cap U_\beta$ of the geodesic distance in G, for which purpose we have fixed an invariant metric on G as in theorem A.2 and remark A.3. In order to apply the subsequent patching lemma 7.2 choose $\delta = \Delta_{\exp} > 0$ to be the radius of a convex geodesic ball in G (as explained before the lemma). Then the u^ν_α can be modified to $u^\nu_\alpha h^\nu_\alpha$ defined on slightly smaller domains $V_\alpha \subset U_\alpha$ that still cover M such that for all $\nu \in \mathbb{N}$ on all intersections $V_\alpha \cap V_\beta$

$$(u^\nu_\alpha h^\nu_\alpha)^{-1}\phi_{\alpha\beta}(u^\nu_\beta h^\nu_\beta) = (h^\nu_\alpha)^{-1}u^\nu_{\alpha\beta}h^\nu_\beta = g_{\alpha\beta} .$$

This also defines no gauge transformation yet, but now we only need to make another ν-independent change: Let $\tilde{u}^\nu_\alpha = u^\nu_\alpha h^\nu_\alpha g_\alpha^{-1}$ on V_α, then this defines a gauge transformation $\tilde{u}^\nu \in \mathcal{G}^{2,p}(P)$ for all $\nu \in \mathbb{N}$. Indeed, $\tilde{u}^\nu$ is defined on $\bigcup_{\alpha=1}^N V_\alpha = M$ and it is well defined since on $V_\alpha \cap V_\beta$

$$(\tilde{u}^\nu_\alpha)^{-1}\phi_{\alpha\beta}\tilde{u}^\nu_\beta = g_\alpha (u^\nu_\alpha h^\nu_\alpha)^{-1}u^\nu_{\alpha\beta}(u^\nu_\beta h^\nu_\beta) g_\beta = g_\alpha g_{\alpha\beta} g_\beta^{-1} = \phi_{\alpha\beta} .$$

For the regularity of $\tilde{u}^\nu$ first note that lemma 7.2 provides $h^\nu_\alpha \in \mathcal{G}^{2,p}(V_\alpha)$. Furthermore, u^ν_α and $g_\alpha = u^1_\alpha$ restricted to V_α lie in $\mathcal{G}^{2,p}(V_\alpha)$ for all α, so the regularity follows from the fact (lemma A.5) that $\mathcal{G}^{2,p}(V_\alpha)$ is closed under group multiplication and inversion.

Finally, we show that $\tilde{u}^\nu_\alpha{}^* A^\nu_\alpha$ is bounded in $\mathcal{A}^{1,p}(V_\alpha)$ for all $\alpha = 1,\dots,N$: The h^ν_α are determined in lemma 7.2 from the $u^\nu_{\alpha\beta}$ that satisfy a uniform bound on $\|(u^\nu_{\alpha\beta})^{-1}\mathrm{d}u^\nu_{\alpha\beta}\|_{W^{1,p}}$. Thus part (ii) of the lemma with $K = 2$ asserts that there also is a uniform bound on $\|(h_\alpha^\nu)^{-1}\mathrm{d}h_\alpha^\nu\|_{W^{1,p}}$. Moreover, the $u_\alpha^\nu{}^*A_\alpha^\nu$ are $W^{1,p}$-bounded by the Uhlenbeck gauge as seen above, so lemma A.6 implies that $h_\alpha^\nu{}^*u_\alpha^\nu{}^*A_\alpha^\nu$ is $W^{1,p}$-bounded as well. Lastly, $(g_\alpha^{-1})^*$ is a ν-independent continuous map on $\mathcal{A}^{1,p}(V_\alpha)$ (again by lemma A.6)[1] , so for every α we obtain a $W^{1,p}$-bound on $(g_\alpha^{-1})^*h_\alpha^\nu{}^*u_\alpha^\nu{}^*A_\alpha^\nu = \tilde{u}_\alpha^\nu{}^*A_\alpha^\nu$.

Now the Banach Alaoglu theorem B.4 asserts that for every α the sequence $\tilde{u}_\alpha^\nu{}^*A_\alpha^\nu$ has a $W^{1,p}$-weakly convergent subsequence. This can be chosen the same subsequence $(\nu_i)_{i\in\mathbb{N}}$ for all $\alpha = 1,\dots,N$. (This could even be done by a diagonal subsequence if there were countably many α.) Then the sequence $\tilde{u}^{\nu_i}A^{\nu_i}$ converges $W^{1,p}$-weakly on all of M. □

This proves the weak Uhlenbeck compactness for compact base manifolds up to the actual patching, which we have postponed to the subsequent technical lemma. Here we do not assume M to be compact, but still G is a compact Lie group equipped with an invariant metric as in theorem A.2.

For the patching arguments let $\Delta_{\exp} > 0$ be the radius of a **convex geodesic ball** $B_{\Delta_{\exp}}(\mathbb{1}) \subset G$ around $\mathbb{1}$ with the following two properties: Firstly, the exponential map is a bijection between $B_{\Delta_{\exp}}(0) \subset \mathfrak{g}$ and $B_{\Delta_{\exp}}(\mathbb{1})$. Secondly, for all $g, h \in B_{\Delta_{\exp}}(\mathbb{1})$ there is a unique minimal geodesic from g to h and this lies within $B_{\Delta_{\exp}}(\mathbb{1})$. For the existence of such balls see e.g. [GHL, 2.89,2.90]. Moreover, since the left multiplications are isometries of G there exist convex geodesic balls $B_{\Delta_{\exp}}(g)$ of the same radius around all $g \in G$.

Furthermore, for two continuous maps u and v from some domain U to G we denote by

$$\mathrm{d}(u,v) := \sup_{x\in U} \mathrm{d}_G(u(x), v(x))$$

the supremum over their domain of the geodesic distance d_G between the values in G. This metric is invariant under left and right multiplication by continuous maps, see appendix A.

We then consider transition functions on open coverings $M = \bigcup_{\alpha\in\mathbb{N}} U_\alpha$ by precompact open sets U_α, that is $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ that satisfy the cocyle conditions as in (A.1),

$$g_{\alpha\alpha} \equiv \mathbb{1} \quad \text{and} \qquad g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{7.1}$$

In particular, note that $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$ on $U_\alpha \cap U_\beta$.

The following patching lemma is stated for infinite coverings but also applies to finite coverings, in which case we just set $U_\alpha = \varnothing$ for $\alpha > N$. In this chapter, the bounds in (ii) will only be needed in the case $K = 2$. The more general formulation, in particular $K = \infty$, can be used to prove the strong compactness theorem E.

[1] If one wants to prove the strong compactness theorem E along the same line of argument, then this point requires a slight modification: Instead of $g_\alpha := u_\alpha^1$ one has to choose a smooth g_α that is $\mathcal{C}^0$-close to u_α^1. This ensures that $g_{\alpha\beta}$ is smooth and that the transformation by the g_α preserves all $W^{k,p}$-bounds.

Lemma 7.2 *Let M be an n-manifold and let $p > \frac{n}{2}$. Fix a locally finite open covering $M = \bigcup_{\alpha\in\mathbb{N}} U_\alpha$ by precompact sets U_α. Then one can find open subsets $V_\alpha \subset U_\alpha$ still covering M such that the following holds:*

(i) Let $k \in \mathbb{N}$ and let $g_{\alpha\beta}, h_{\alpha\beta} \in \mathcal{G}^{k+1,p}(U_\alpha \cap U_\beta)$ be two sets of transition functions that satisfy (7.1) and

$$\mathrm{d}(g_{\alpha\beta}, h_{\alpha\beta}) \leq \Delta_{\exp} \qquad \forall \alpha, \beta \in \mathbb{N}.$$

Then there exist local gauge transformations $h_\alpha \in \mathcal{G}^{k+1,p}(V_\alpha)$ for all $\alpha \in \mathbb{N}$ such that on all intersections $V_\alpha \cap V_\beta$

$$h_\alpha^{-1} h_{\alpha\beta} h_\beta = g_{\alpha\beta}. \tag{7.2}$$

(ii) Let the $h_{\alpha\beta}$ in (i) run through a sequence $h^\nu_{\alpha\beta}$ of sets of transition functions such that $g_{\alpha\beta}, h^\nu_{\alpha\beta} \in \mathcal{G}^{k+1,p}(U_\alpha \cap U_\beta)$ for all $k < K$, where $K \geq 2$ is an integer or $K = \infty$. Assume that for every $\alpha, \beta \in \mathbb{N}$ and $k < K$ there is a uniform bound on $\|(h^\nu_{\alpha\beta})^{-1}\mathrm{d}h^\nu_{\alpha\beta}\|_{W^{k,p}(U_\alpha\cap U_\beta)}$.
Then the gauge transformations h^ν_α in (i) are constructed in such a way that for every $\alpha \in \mathbb{N}$ and $k < K$ they satisfy $h^\nu_\alpha \in \mathcal{G}^{k+1,p}(V_\alpha)$ and

$$\sup_{\nu\in\mathbb{N}} \|(h^\nu_\alpha)^{-1}\mathrm{d}h^\nu_\alpha\|_{W^{k,p}(V_\alpha)} < \infty.$$

Proof: We construct the V_α and h_α inductively. It will be clear from the local finiteness of the covering that this is a finite construction for every fixed α (in the j-th step V_α and h_α remain unchanged if $U_\alpha \cap U_j = \emptyset$). Moreover, the V_α are constructed independently of $k \in \mathbb{N}$ and the transition functions $g_{\alpha\beta}$ and $h_{\alpha\beta}$.

For $j = 1$ set $V_1 := U_1$ and $h_1 := \mathbb{1}$. Then note that for all $i \geq 2$ we have $\mathrm{d}(h_{i1}h_1g_{1i}, \mathbb{1}) = \mathrm{d}(g_{1i}, h_{1i}) \leq \Delta_{\exp}$ on $V_1 \cap U_i$ by assumption. Now for some $j \geq 2$ assume that we have the following construction for $\alpha \leq j-1$ (this is the case for $j = 2$) and then obtain the analogous situation for all $\alpha \leq j$ by the subsequent construction for $\alpha = j$.

Construction for $\alpha \leq j-1$:

For all $\alpha \leq j-1$ we have constructed an open set $V_\alpha \subset U_\alpha$ and $h_\alpha \in \mathcal{G}^{k+1,p}(V_\alpha)$ such that $M = \bigcup_{\alpha\geq j} U_j \cup \bigcup_{\alpha<j} V_\alpha$ and (7.2) holds for all $\alpha, \beta \leq j-1$. Moreover, for all $\alpha \leq j-1$ and $i \geq j$ on $V_\alpha \cap U_i$

$$\mathrm{d}(h_{i\alpha}h_\alpha g_{\alpha i}, \mathbb{1}) \leq \Delta_{\exp}. \tag{7.3}$$

Construction for $\alpha = j$:

First note that (7.2) is trivially satisfied for $\alpha = \beta = j$ no matter how we define h_j. Secondly, if (7.2) is satisfied for some pair (α, β) then it also holds for (β, α):

$$h_\beta^{-1} h_{\beta\alpha} h_\alpha = h_\beta^{-1}(h_\alpha^{-1}h_{\alpha\beta})^{-1} = h_\beta^{-1}(g_{\alpha\beta}h_\beta^{-1})^{-1} = g_{\beta\alpha}.$$

Thus the task is to define h_j on $V_j := U_j$ such that (7.2) is satisfied for $\alpha \leq j-1$ and $\beta = j$. This prescribes h_j on the intersection of U_j with all V_α for $\alpha \leq j-1$. So consider the map $\rho_j : U_j \cap \bigcup_{\alpha<j} V_\alpha \to G$ given by $h_{j\alpha} h_\alpha g_{\alpha j}$ on $U_j \cap V_\alpha$. It is well defined since on all intersections $U_j \cap V_\alpha \cap V_\beta$

$$h_{j\alpha} h_\alpha g_{\alpha j} = h_{j\beta} h_{\alpha\beta}^{-1} h_\alpha g_{\alpha\beta} g_{\beta j} = h_{j\beta} h_\beta g_{\beta j}.$$

Moreover, ρ_j is a $\mathcal{G}^{k+1,p}$-map on its domain since this Sobolev space is closed under group multiplication (lemma A.5), and all factors are $\mathcal{G}^{k+1,p}$-maps by assumption and the construction for $\alpha \leq j-1$. Also by the previous construction ρ_j takes values in the convex geodesic ball $B_{\Delta_{\exp}}(\mathbb{1})$. Hence it can be composed with the inverse of the exponential map on G to define $\xi_j \in W^{k+1,p}(U_j \cap \bigcup_{\alpha<j} V_\alpha, \mathfrak{g})$ such that for all $\alpha \leq j-1$

$$h_{j\alpha} h_\alpha g_{\alpha j} = \rho_j = \exp(\xi_j) \qquad \text{on } U_j \cap V_\alpha.$$

To see that indeed, ξ_j is a $W^{k+1,p}$-function, note that composition with the exponential map is a homeomorphism with respect to the $W^{k+1,p}$-topologies on $\mathfrak{g}$ and G by the definition of the $\mathcal{G}^{k+1,p}$-topology in lemma B.7.

The requirement from (7.2) on h_j is to equal $\exp(\xi_j)$ on $U_j \cap \bigcup_{\alpha<j} V_\alpha$. Here we are allowed to replace the V_α that intersect U_j by some smaller open subsets such that M is still covered.

Denote $N := M \setminus \bigcup_{\alpha \geq j} U_\alpha$. This is closed and covered by finitely many precompact sets $U_1, \ldots, U_{j-1}$, hence N is compact. Now going through $\ell = 1, \ldots, j-1$ we will replace V_ℓ by $V'_\ell \subset V_\ell$ such that the closure C of $U_j \cap \bigcup_{\alpha<j} V'_\alpha$ (where we will define $h_j := \exp(\xi_j)$) becomes disjoint from $B := \overline{U}_j \setminus \bigcup_{\alpha<j} V_\alpha$ (where we will have to define $h_j \equiv \mathbb{1}$). The covering is to be preserved in each step, i.e. for $\ell = 1, \ldots, j-1$

$$N \subset \bigcup_{\alpha \leq \ell} V'_\alpha \cup \bigcup_{\ell<\alpha<j} V_\alpha. \tag{7.4}$$

In order to define such V'_ℓ consider

$$\begin{aligned} A_\ell &:= N \setminus \Big(\bigcup_{\alpha<\ell} V'_\alpha \cup \bigcup_{\ell<\alpha<j} V_\alpha \Big), \\ B_\ell &:= \overline{U}_j \setminus V_\ell. \end{aligned}$$

These are compact sets and $A_\ell \subset V_\ell \subset {B_\ell}^{\complement}$. Thus there exists an open $V'_\ell \subset V_\ell$ such that $A_\ell \subset V'_\ell \subset \overline{V'_\ell} \subset {B_\ell}^{\complement}$. Note that in the case $V_\ell \cap U_j = \varnothing$ we have $B_\ell = \varnothing$ and thus simply define $V'_\ell := V_\ell$ (this makes the overall construction finite). This construction is shown in figure 5.1.

The covering requirement (7.4) is met due to $A_\ell \subset V'_\ell$. Moreover, $\overline{V'_\ell}$ and B_ℓ are disjoint for all $\ell < j$. Hence the intersection of $C \subset \bigcup_{\ell<j} \overline{V'_\ell}$ and $B = \bigcap_{\ell<j} B_\ell$ also is empty. Indeed, an element of the intersection would have to lie in some $\overline{V'_\ell}$ but also in B_ℓ.

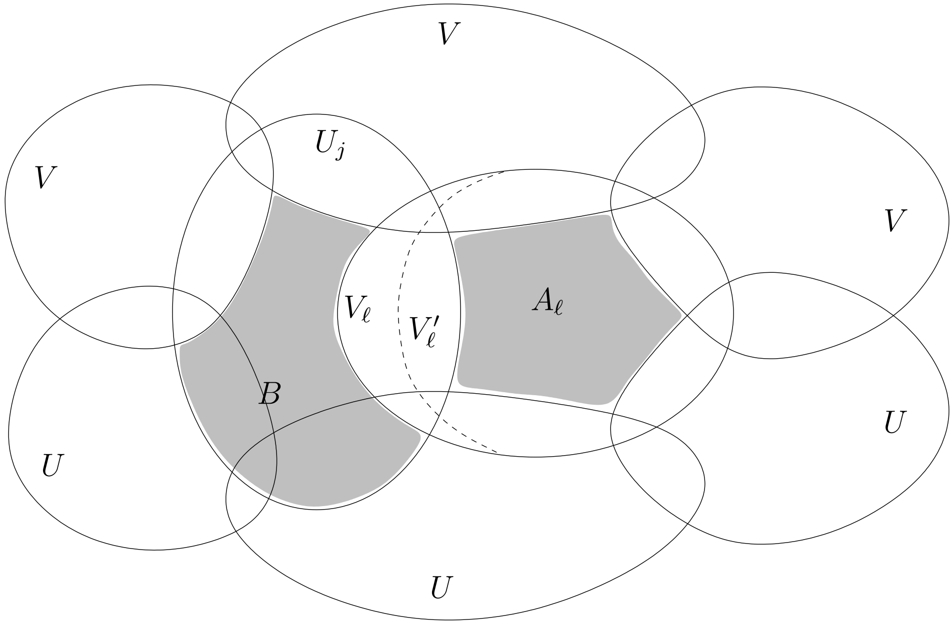

Figure 8.1: Construction of V'_ℓ

After the compact sets B and C have been made disjoint there exists a smooth cutoff function $\psi_j : \overline{U}_j \to [0,1]$ that equals 0 on B and 1 on C. (Note that this function is chosen independently of the transition functions.) Now on $V_j = U_j$ define $h_j := \exp(\psi_j \xi_j)$, then

$$h_j \begin{cases} = h_{j\alpha} h_\alpha g_{\alpha j} & \text{on } V_j \cap V'_\alpha \text{ for } \alpha \le j-1, \\ \in \gamma(\mathbb{1}, h_{j\alpha} h_\alpha g_{\alpha j}) & \text{on } V_j \cap V_\alpha \text{ for } \alpha \le j-1, \\ \equiv \mathbb{1} & \text{on } V_j \setminus \bigcup_{\alpha<j} V_\alpha, \end{cases}$$

where $\gamma(\mathbb{1}, h)$ denotes the unique minimal geodesic between $\mathbb{1}$ and $h \in B_{\Delta_{\exp}}(\mathbb{1})$. The first equality shows that (7.2) holds for all $\alpha, \beta \le j$ if we replace the V_α by V'_α. The other two properties of h_j help to establish (7.3): For $\alpha \le j-1$ and $i \ge j+1$ this holds on $V_\alpha \cap U_i \supset V'_\alpha \cap U_i$ by the previous construction. For $\alpha = j$ and $i \ge j+1$ we also check this on $V_j \cap U_i$. Firstly, on $V_j \cap U_i \setminus \bigcup_{\alpha<j} V_\alpha$ this simply follows from the assumption and $h_j \equiv \mathbb{1}$,

$$\mathrm{d}(h_{ij} h_j g_{ji}, \mathbb{1}) = \mathrm{d}(h_{ij} g_{ji}, \mathbb{1}) = \mathrm{d}(g_{ji}, h_{ji}) \le \Delta_{\exp}.$$

On $V_j \cap U_i \cap V_\alpha$ for some $\alpha \le j-1$ one also has

$$\mathrm{d}(h_{ij} h_j g_{ji}, \mathbb{1}) = \mathrm{d}(h_j, h_{ji} g_{ij}) \le \Delta_{\exp}$$

since h_j lies in the convex geodesic ball $B_{\Delta_{\exp}}(h_{ji}g_{ij})$. Indeed, h_j lies on the unique minimal geodesic between $\mathbb{1}$ and $h_{j\alpha}h_\alpha g_{\alpha j}$, and this entirely lies in $B_{\Delta_{\exp}}(h_{ji}g_{ij})$ since both endpoints do so: Use the assumption, the cocycle identities (7.1), and the construction for $\alpha \leq j-1$ to check that

$$\mathrm{d}(\mathbb{1}, h_{ji}g_{ij}) \;=\; \mathrm{d}(h_{ij}, g_{ij}) \;\leq\; \Delta_{\exp},$$

$$\mathrm{d}(h_{j\alpha}h_\alpha g_{\alpha j}, h_{ji}g_{ij}) \;=\; \mathrm{d}(h_\alpha, h_{\alpha j}h_{ji}g_{ij}g_{j\alpha}) \;=\; \mathrm{d}(h_\alpha, h_{\alpha i}g_{i\alpha}) \;\leq\; \Delta_{\exp}.$$

Now (7.3) is established and it remains to check the regularity. For $\alpha \leq j-1$ the h_α are only restricted to smaller domains, so still $h_\alpha \in \mathcal{G}^{k+1,p}(V'_\alpha)$. Also $h_j = \exp(\psi_j\xi_j) \in \mathcal{G}^{k+1,p}(V_j)$ by the definition in lemma B.5 since ψ_j is smooth and thus $\psi_j\xi_j \in W^{k+1,p}(V_j, \mathfrak{g})$. Thus the construction can be continued for $\alpha = j+1$ with the new $V_\alpha := V'_\alpha$. That way for all $\alpha \in \mathbb{N}$ we define V_α and h_α by finitely many steps, and this proves (i).

The regularity of the h^ν_α in (ii) simply follows from the fact that their above construction is independent of the Sobolev index k. The uniform bounds for every $k < K$ in (ii) can also be established by induction over α. The start is trivial since $h^\nu_1 \equiv \mathbb{1}$. Then as induction hypothesis assume that for all $\alpha \leq j-1$ the norms $\|(h^\nu_\alpha)^{-1}\mathrm{d}h^\nu_\alpha\|_{W^{k,p}(U_\alpha)}$ are uniformly bounded. (Note that we use the domain U_α on which the h^ν_α are defined originally before the domain is cut down to V_α in the subsequent steps of the construction.) For the induction step $\alpha = j$ recall that $h^\nu_j = \exp(\psi_j\xi^\nu_j)$ on U_j, where $\xi^\nu_j = \exp^{-1}(\rho^\nu_j)$ is given by $\rho^\nu_j = h^\nu_{j\alpha}h^\nu_\alpha g_{\alpha j}$ on $U_j \cap V_\alpha$. There are uniform bounds on $\|(h^\nu_{j\alpha})^{-1}\mathrm{d}h^\nu_{j\alpha}\|_{W^{k,p}}$ by assumption, on $\|(h^\nu_\alpha)^{-1}\mathrm{d}h^\nu_\alpha\|_{W^{k,p}}$ by the induction hypothesis, and on $\|(g_{j\alpha})^{-1}\mathrm{d}g_{j\alpha}\|_{W^{k,p}}$ since this is independent of ν. These uniform bounds provide a uniform bound on $\|(\rho^\nu_j)^{-1}\mathrm{d}\rho^\nu_j\|_{W^{k,p}}$ as the following calculation shows:

For the precompact subset $U := U_j \cap V_\alpha \subset M$ there exists a constant C such that for all $u, v \in \mathcal{G}^{k+1,p}(U)$

$$\begin{aligned}\|(uv)^{-1}\mathrm{d}(uv)\|_{W^{k,p}} &\leq \|v^{-1}(u^{-1}\mathrm{d}u)v\|_{W^{k,p}} + \|v^{-1}\mathrm{d}v\|_{W^{k,p}} \\ &\leq C\|u^{-1}\mathrm{d}u\|_{W^{k,p}}\bigl(1 + \|v^{-1}\mathrm{d}v\|_{W^{k,p}}\bigr)^k + \|v^{-1}\mathrm{d}v\|_{W^{k,p}}.\end{aligned}$$

This estimate uses lemma A.7 and the Sobolev estimate for $W^{k,p} \hookrightarrow W^{k-1,2p}$. We apply this firstly to $(u,v) = (h^\nu_{j\alpha}, h^\nu_\alpha)$ and then to $(u,v) = (h^\nu_{j\alpha}h^\nu_\alpha, g_{\alpha j})$ to obtain the uniform bounds for $\rho^\nu_j = h^\nu_{j\alpha}h^\nu_\alpha g_{\alpha j}$.

Now choose the exponential map as a chart of $B_{\Delta_{\exp}}(\mathbb{1})$ in which all ρ^ν_j take their values. Then a $W^{k+1,p}$-bound on ξ^ν_j follows analogously to the equivalence of (i) and (iv) in lemma B.5 (see the calculations for (iv)⇒(ii), then from the estimates for an embedding in (ii) one obtains estimates for some chart and thus for all charts by the equivalence of the respective norms). Furthermore, ψ_j is a ν-independent smooth function on the closure $\overline{U}_j$, thus $\psi_j\xi^\nu_j$ is $W^{k+1,p}$-bounded, too. Finally, by the calculations for (iii)⇒(iv) in lemma B.5 we obtain a uniform bound on $\|(h^\nu_j)^{-1}\mathrm{d}h^\nu_j\|_{W^{k,p}(U_j)}$. □

In the application for the proof of theorem 7.1 the existence of the h_α^ν is not surprising; $h_\alpha^\nu = (u_\alpha^\nu)^{-1} g_\alpha$ also meets (7.2). But for these functions there is no uniform bound on $(h_\alpha^\nu)^{-1}\mathrm{d}h_\alpha^\nu$. In other words, it is already clear that the transition functions are describing the same bundle, but one has to find the right bundle isomorphism that provides the weak convergence of the connections.

More generally, the lemma says that two sets of transition functions are describing the same bundle if they are sufficiently $\mathcal{C}^0$-close (only depending on the Lie group). This is because (7.2) is just the condition in lemma A.1.

In the generalization of weak Uhlenbeck compactness to noncompact manifolds (theorem A′) one considers a noncompact Riemannian n-manifold $M = \bigcup_{k\in\mathbb{N}} M_k$ that is exhausted by compact submanifolds $M_k \subset M$ such that each M_k is contained in the interior of M_{k+1} for all $k \in \mathbb{N}$. Here the interior of a subset $N \subset M$ is defined with respect to the relative topology, $\mathrm{int}(N) = N \setminus \mathrm{cl}(M \setminus N)$. As before let $1 < p < \infty$ be such that $p > \frac{n}{2}$. Then $\mathcal{G}^{2,p}(P|_{M_k})$ and its action on $\mathcal{A}^{1,p}(P|_{M_k})$ are well defined for all $k \in \mathbb{N}$. (Note that we need a compact base manifold for the definition of the $W^{2,p}$-space of gauge transformations.)

We could now proceed as in the proof for the compact case and obtain a locally finite open covering $M = \bigcup_{\alpha\in\mathbb{N}} U_\alpha$ and gauge transformations $\tilde{u}_\alpha$ on all U_α that put the given connections into Uhlenbeck gauge. Now there is only one thing that stops us from applying the patching lemma 7.2 and deduce the weak convergence of a diagonal subsequence over all M_k. This is the condition $\mathrm{d}(g_{\alpha\beta}, h_{\alpha\beta}) \leq \Delta_{\exp}$ that has to be met for all $\alpha, \beta \in \mathbb{N}$. If for some reason the convergence of the transition functions $u_{\alpha\beta}^\nu$ was uniform with respect to $\alpha, \beta \in \mathbb{N}$ then the weak Uhlenbeck compactness would straight away generalize to exhausted manifolds without any topological obstructions. However, there seem to be obstructions to the Uhlenbeck compactness when the nontrivial topology of a noncompact manifold M is not restricted to a compact submanifold M_1. This is why we need the additional assumption that the exhausting manifolds are deformation retracts of M. We make this precise by the following definition.

Definition 7.3 *A subset X of a smooth manifold M is called a* **deformation retract** *of M if there exists a continuous map $\Phi : [0,1] \times M \to M$ such that*

$$\Phi(0,\cdot) = \mathrm{Id}_M, \qquad \Phi(1,M) \subset X, \qquad \textit{and} \quad \Phi(t,\cdot)|_X = \mathrm{Id}_X \quad \forall t \in [0,1].$$

Remark 7.4 Let $X \subset M$ be a compact submanifold and a deformation retract of M, and let $\Omega \subset \mathrm{int}(X)$ be a compact subset. Then there exists a smooth map $\Psi : [0,1] \times M \to M$ such that

$$\Psi(0,\cdot) = \mathrm{Id}_M, \qquad \Psi(1,M) \subset X, \qquad \text{and} \quad \Psi(t,\cdot)|_\Omega = \mathrm{Id}_\Omega \quad \forall t \in [0,1].$$

To see this use a tubular neighbourhood of $\partial X \subset X$ to construct a compact subset $X' \subset \mathrm{int}(X)$ that also is a deformation retract of M such that $\Omega \subset \mathrm{int}(X')$. Let $\Phi \in \mathcal{C}^0([0,1] \times M, M)$ be a continuous retraction to X'. One then finds

$\Psi^\nu \in \mathcal{C}^\infty([0,1]\times M, M)$ that converge to Φ in the strong $\mathcal{C}^0$-topology (see e.g. [H, Thm.2.3.3]). By interpolation to Id_M on X' one can modify the Ψ^ν for sufficiently large ν such that $\Psi^\nu(t,\cdot)|_\Omega = \mathrm{Id}_\Omega$ for all $t\in[0,1]$. [2] By a second interpolation on a neighbourhood of $\{0\}\times M \subset [0,1]\times M$ to Id_M one can moreover achieve $\Psi^\nu(0,\cdot) = \mathrm{Id}_M$. [3] Now Ψ^ν still converges to Φ and $\Phi(1,M)\subset X'$, so for sufficiently large ν one has $\Psi^\nu(1,M)\subset X$. So this is the required smooth map.

Now we can restate theorem A′, that is the weak Uhlenbeck compactness theorem for noncompact manifolds:

Theorem 7.5 *Let $M = \bigcup_{k\in\mathbb{N}} M_k$ be exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Let a sequence $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}_{loc}(P)$ be given and suppose that for all $k\in\mathbb{N}$ there is a uniform bound on $\|F_{A^\nu}\|_{L^p(M_k)}$.*

Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^{\nu\,*}A^\nu|_{M_k}$ weakly converges in $\mathcal{A}^{1,p}(P|_{M_k})$ for all $k\in\mathbb{N}$.

Theorem 7.5 will follow from the weak Uhlenbeck compactness for the compact submanifolds M_k combined with the next proposition (for $I=\{1\}$). This is a general result for sequences of connections and gauge transformations on manifolds that are exhausted by compact deformation retracts. It will again be used to generalize the strong Uhlenbeck compactness to noncompact manifolds. Here we fix a reference connection $\tilde{A}\in\mathcal{A}(P)$ with respect to which the Sobolev norms of connections on P are defined.

Proposition 7.6 *Let $M = \bigcup_{k\in\mathbb{N}} M_k$ be exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M, and let $I=\mathbb{N}$ or $I=\{1,\dots,\ell_0\}$ for some $\ell_0\in\mathbb{N}$. Let a sequence of connections $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}_{loc}(P)$ be given and suppose that the following holds:*

For every k and every subsequence of $(A^\nu)_{\nu\in\mathbb{N}}$ there exist a further subsequence $(\nu_{k,i})_{i\in\mathbb{N}}$ and gauge transformations $u^{k,i}\in\mathcal{G}^{2,p}(P|_{M_k})$ such that

$$\sup_{i\in\mathbb{N}} \left\| u^{k,i\,*}A^{\nu_{k,i}} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall \ell\in I. \tag{7.5}$$

Then there exists a subsequence $(\nu_i)_{i\in\mathbb{N}}$ and a sequence of gauge transformations $u^i\in\mathcal{G}^{2,p}_{loc}(P)$ such that

$$\sup_{i\in\mathbb{N}} \left\| u^{i\,*}A^{\nu_i} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall k\in\mathbb{N}, \ell\in I.$$

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[2] This uses a Whitney embedding $M\hookrightarrow\mathbb{R}^{2n+1}$ and the projection from a tubular neighbourhood in $\mathbb{R}^{2n+1}$ to M, see e.g. [H, Thm.2.2.14, Thm.4.5.1]. If M has a boundary, then one only has a tubular neighbourhood of $\mathrm{int}(M)\subset\mathbb{R}^{2n+1}$. So one first has to perform the interpolation in a tubular neighbourhood of $\partial M\subset M$ using an embedding $\partial M\hookrightarrow\mathbb{R}^{2n-1}$. Then a second interpolation can be performed in the interior of M.

[3] In the first interpolation one had $\Psi^\nu\to\mathrm{Id}|_M$ on $[0,1]\times X'$. Here again Ψ^ν comes sufficiently close to $\mathrm{Id}|_M$ for the interpolation: First chooses the neighbourhood of $\{0\}\times M$ such that Φ is suitably close to $\mathrm{Id}|_M$, then use the strong $\mathcal{C}^0$-convergence of the Ψ^ν.

The key to this proposition is an extension argument given by Donaldson and Kronheimer [DK, Lemma 4.4.5], which we explain in lemma 7.8 below. Here the crucial point is to be able to extend a gauge transformation defined on a compact subset to the whole manifold. This is ensured by the assumption that the exhausting compact submanifolds are deformation retracts of M, as will be shown in the following auxiliary lemma.

Lemma 7.7 *Let $K \subset M$ be a compact submanifold and deformation retract of M and let $\Omega \subset \mathrm{int}(K)$ be a compact subset. Then the following holds.*

(i) For every smooth gauge transformation $u \in \mathcal{G}(P|_K)$ there exists a smooth extension $\tilde{u} \in \mathcal{G}(P)$ such that $\tilde{u}|_\Omega = u|_\Omega$.

(ii) For every gauge transformation $u \in \mathcal{G}^{2,p}(P|_K)$ there exists a gauge transformation $\tilde{u} \in \mathcal{G}^{2,p}_{loc}(P)$ such that $\tilde{u}|_\Omega = u|_\Omega$.

Proof: Fix a smooth connection of $P \times_c G$ (induced by a smooth connection of P). Let $\Psi : [0,1] \times M \to M$ be a smooth retraction of M to K with $\Psi(t,\cdot)|_\Omega = \mathrm{Id}|_\Omega$ for all $t \in [0,1]$ as in remark 7.4. Then for any given smooth gauge transformation $u : K \to (P \times_c G)|_K$ we define $\tilde{u} : M \to P \times_c G$ as follows: For every $x \in M$ let $\tilde{u}(x) \in (P \times_c G)_x$ be the parallel transport of $u(\Psi(1,x)) \in (P \times_c G)_{\Psi(1,x)}$ along the path $\Psi(\cdot,x)$ from $\Psi(1,x) \in K$ to $\Psi(0,x) = x$. For $x \in \Omega$ this construction provides $\tilde{u}(x) = u(x)$ since in that case $\Psi(\cdot,x) \equiv x$. So we obtain a well-defined extension $\tilde{u}$ of u. Moreover, parallel transport along smooth paths is a smooth map, hence $\tilde{u} \in \mathcal{G}(P)$ is smooth. This proves (i).

For (ii) let $u \in \mathcal{G}^{2,p}(P|_K)$ be given. Then by the definition of this Sobolev space there is a smooth gauge transformation $s \in \mathcal{G}(P|_K)$ and some $\xi \in W^{2,p}(K, \mathfrak{g}_P)$ such that $u = s \cdot \exp(\xi)$. Use (i) to vary s outside of Ω and extend it to $s \in \mathcal{G}(P)$. Since $\Omega \subset \mathrm{int}K$ one also finds a smooth cutoff function $h \in \mathcal{C}^\infty(M,[0,1])$ that equals 1 on Ω and vanishes on $M \backslash K$. Use this to define $\tilde{u} := s \cdot \exp(h\xi)$ on all of M. This gauge transformation clearly satisfies $\tilde{u}|_\Omega = u|_\Omega$. Moreover $\tilde{u}|_K \in \mathcal{G}^{2,p}(P|_K)$ and $\tilde{u}|_{M\backslash K} = s|_{M\backslash K}$ is smooth, hence $\tilde{u} \in \mathcal{G}^{2,p}_{loc}(P)$ is the required extension. □

This lemma in fact also holds with $\Omega = K$ since K has a smooth boundary, but this would require the use of Sobolev extension operators, see e.g. [A, Thm.4.26].

Lemma 7.8 (Donaldson, Kronheimer)
Let $\Omega \subset M'' \subset M'$ be compact submanifolds of M such that $\Omega \subset \mathrm{int}(M'')$ and M'' is a deformation retract of M. Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P|_{M'})$ be a sequence of connections, and let $(u^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{G}^{2,p}(P|_{M''})$ and $(v^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{G}^{2,p}(P|_{M'})$ be sequences of gauge transformations such that both $\|u^{\nu\,*}A^\nu\|_{W^{1,p}(M'')}$ and $\|v^{\nu\,*}A^\nu\|_{W^{1,p}(M')}$ are uniformly bounded.

Then after passing to a subsequence (again labelled by $\nu \in \mathbb{N}$) one can find gauge transformations $w^\nu \in \mathcal{G}^{2,p}(P|_{M'})$ such that $w^\nu|_\Omega = u^\nu|_\Omega$ and $\|w^{\nu\,*}A^\nu\|_{W^{1,p}(M')}$ is uniformly bounded.

If moreover for some $\ell \in \mathbb{N}$ both $\|u^{\nu\,*}A^\nu\|_{W^{\ell,p}(M'')}$ and $\|v^{\nu\,*}A^\nu\|_{W^{\ell,p}(M')}$ are uniformly bounded, then $\|w^{\nu\,*}A^\nu\|_{W^{\ell,p}(M')}$ also is uniformly bounded.

Proof: Consider the gauge transformations $(v^\nu)^{-1}u^\nu \in \mathcal{G}^{2,p}(P|_{M''})$ (their regularity follows from lemma A.5). They transform $v^{\nu\,*}A^\nu$ into $u^{\nu\,*}A^\nu$, which both are bounded sequences in $\mathcal{A}^{1,p}(P|_{M''})$. Thus lemma A.8 provides a subsequence (that we again label by $\nu \in \mathbb{N}$) and a limit $h^\infty \in \mathcal{G}^{2,p}(P|_{M''})$ such that $\mathrm{d}((v^\nu)^{-1}u^\nu, h^\infty) \leq \frac{1}{2}\Delta_{\exp}$ for all $\nu \in \mathbb{N}$. Here the gauge transformations have to be considered as equivariant maps from $P|_{M''}$ to G, so d denotes the supremum over $P|_{M''}$ of the geodesic distances in G. Then let $\Delta_{\exp} > 0$ be the injectivity radius of the exponential map on G.

Now we can choose a smooth gauge transformation $h \in \mathcal{G}(P|_{M''})$ such that $\mathrm{d}(h, h^\infty) \leq \frac{1}{2}\Delta_{\exp}$ (it exists since gauge transformations are sections of the smooth bundle $P \times_c G$). Then the gauge transformations $h^{-1}(v^\nu)^{-1}u^\nu \in \mathcal{G}^{2,p}(P|_{M''})$ only take values in the $\Delta_{\exp}$-geodesic ball around $\mathbb{1}$ (their regularity is due to lemma A.5). These can be composed with the inverse exponential map to obtain sections $\xi^\nu \in W^{2,p}(M'', \mathfrak{g}_P)$. (For the regularity use lemma B.8 and the fact that $\exp : \mathfrak{g} \to G$ is smooth.)

Lemma 7.7 (i) provides $\tilde{h} \in \mathcal{G}(P)$ such that $\tilde{h}|_\Omega = h|_\Omega$. All derivatives of $\tilde{h}|_{M'}$ are bounded (in the bundle charts), hence $\tilde{h} \in \mathcal{G}^{2,p}(P|_{M'})$. Choose a smooth cutoff function $\psi : M' \to [0,1]$ that equals 1 on Ω and vanishes on $M' \setminus M''$, and then define

$$w^\nu := v^\nu \tilde{h}\, \exp(\psi{\circ}\pi \cdot \xi^\nu).$$

To see that this defines a gauge transformation in $\mathcal{G}^{2,p}(P|_{M'})$ first note that this space is closed under mulitplication by lemma A.5. So since the v^ν and $\tilde{h}$ lie in that space it suffices to examine the last factor. Here $\psi{\circ}\pi$ is an equivariant cutoff function that is multiplied with the equivariant maps $\xi^\nu : P|_{M''} \to \mathfrak{g}$ to define an equivariant map from $P|_{M'}$ to $\mathfrak{g}$. This map $\psi{\circ}\pi \cdot \xi^\nu$ is of class $W^{2,p}$, hence composition with the exponential map yields a gauge transformation in $\mathcal{G}^{2,p}(P|_{M'})$. This establishes the required regularity of w^ν. The construction moreover satisfies

$$w^\nu|_\Omega \;=\; v^\nu h \exp(1 \cdot \xi^\nu) \;=\; v^\nu h h^{-1}(v^\nu)^{-1}u^\nu \;=\; u^\nu.$$

Now assume uniform bounds on $\|u^{\nu\,*}A^\nu\|_{W^{\ell,p}(M'')}$ and $\|v^{\nu\,*}A^\nu\|_{W^{\ell,p}(M')}$ for some $\ell \in \mathbb{N}$, possibly $\ell = 1$. Then in every local trivialization over $U_\alpha \subset M''$ lemma A.8 provides a uniform bound on $\big\|\big((v^\nu_\alpha)^{-1}u^\nu_\alpha\big)\mathrm{d}\big((v^\nu_\alpha)^{-1}u^\nu_\alpha\big)\big\|_{W^{\ell,p}(M'')}$. Next, in analogy to the equivalence of (iii) and (iv) in lemma B.5 one obtains uniform bounds on $\xi^\nu_\alpha = \exp^{-1}(h_\alpha^{-1}(v^\nu_\alpha)^{-1}u^\nu_\alpha) \in W^{\ell+1,p}(U_\alpha, \mathfrak{g})$. (See the calculations for (iv)⇒(ii), then from the estimates for an embedding in (ii) one obtains estimates in some chart and thus in all charts by the equivalence of the respective norms). When multiplied by the smooth cutoff function ψ the local representatives of ξ^ν extend to uniformly bounded sequences in $W^{\ell+1,p}(U_\alpha, \mathfrak{g})$ for every local trivialization over $U_\alpha \subset M'$ (equalling zero outside of M''). Then composition with the exponential map yields gauge transformations $g^\nu := \exp(\psi{\circ}\pi \cdot \xi^\nu) \in \mathcal{G}^{\ell+1,p}(P|_{M'})$. Moreover, $\|(g^\nu_\alpha)^{-1}\mathrm{d}g^\nu_\alpha\|_{W^{\ell,p}}$ is uniformly bounded in every local trivialization over $U_\alpha \subset M'$ by the calculations for (iii)⇒(iv) in lemma B.5.

To establish the uniform bound we rewrite $w^{\nu\,*}A^{\nu} = g^{\nu\,*}h^{*}v^{\nu\,*}A^{\nu}$. By assumption $\|v^{\nu\,*}A^{\nu}\|_{W^{\ell,p}(M')}$ is uniformly bounded, then $\|h^{*}v^{\nu\,*}A^{\nu}\|_{W^{\ell,p}(M')}$ also is uniformly bounded since h^{*} is a continuous map on $\mathcal{A}^{\ell,p}(P|_{M'})$. The uniform bound on $\|w^{\nu\,*}A^{\nu}\|_{W^{\ell,p}}$ then follows from the local estimate in lemma A.6. To see this cover M' with finitely many bundle charts U_α, then firstly one obtains uniform bounds on the local representatives of $w^{\nu\,*}A^{\nu}$: For some constants C_α

$$\begin{aligned}\|(w^{\nu\,*}A^{\nu})_\alpha\|_{W^{\ell,p}} \leq \|(g^{\nu}_\alpha)^{-1}\mathrm{d}g^{\nu}_\alpha\|_{W^{\ell,p}}& \\ + C_\alpha\|(h^{*}v^{\nu\,*}A^{\nu})_\alpha\|_{W^{\ell,p}}&\bigl(1+\|(g^{\nu}_\alpha)^{-1}\mathrm{d}g^{\nu}_\alpha\|_{W^{\ell-1,2p}}\bigr)^{\ell}.\end{aligned}$$

Here the right hand side is uniformly bounded due to the uniform bounds on $\|(g^{\nu})^{-1}\mathrm{d}g^{\nu}\|_{W^{\ell,p}}$ and $\|h^{*}v^{\nu\,*}A^{\nu}\|_{W^{\ell,p}}$, and by the estimate for $W^{\ell,p}\hookrightarrow W^{\ell-1,2p}$. For these local Sobolev norms of connections the reference connection is defined from the trivialization of the bundle, but we defined the global Sobolev norm with respect to the fixed reference connection $\tilde{A}$. However, $\sum_\alpha \|\tilde{A}_\alpha\|_{W^{\ell,p}}$ is finite since we chose finitely many bundle charts. Hence we obtain a global uniform bound

$$\begin{aligned}\|w^{\nu\,*}A^{\nu}\|_{W^{\ell,p}} &= \bigl\|w^{\nu\,*}A^{\nu}-\tilde{A}\bigr\|_{W^{\ell,p}(M,\mathrm{T}^{*}M\otimes\mathfrak{g}_P)} \\ &\leq \sum_\alpha \bigl\|(w^{\nu\,*}A^{\nu})_\alpha-\tilde{A}_\alpha\|_{W^{\ell,p}(U_\alpha,\mathrm{T}^{*}U_\alpha\otimes\mathfrak{g})} \\ &\leq \sum_\alpha \|\tilde{A}_\alpha\|_{W^{\ell,p}(U_\alpha,\mathrm{T}^{*}U_\alpha\otimes\mathfrak{g})} + \sum_\alpha \|(w^{\nu\,*}A^{\nu})_\alpha\|_{W^{\ell,p}(U_\alpha,\mathrm{T}^{*}U_\alpha\otimes\mathfrak{g})}.\end{aligned}$$

□

Proof of proposition 7.6 :
Let a sequence of connections $A^{\nu}\in\mathcal{A}^{1,p}_{loc}(P)$ be given as supposed. In order to find a gauge for a subsequence such that it is bounded on all M_k one can not simply choose a diagonal subsequence but also has to define the gauge transformations independently of k. For that purpose we first construct inductively for all $k\in\mathbb{N}$ a subsequence $(\mu_{k,i})_{i\in\mathbb{N}}$ and gauge transformations $w(k,\mu_{k,i})\in\mathcal{G}^{2,p}_{loc}(P)$ such that the following holds:

(i) $(\mu_{j,i})_{i\in\mathbb{N}}\subset(\mu_{k,i})_{i\in\mathbb{N}} \qquad \forall j>k$,

(ii) $w(j,\mu_{j,i})|_{M_k} = w(k,\mu_{j,i})|_{M_k} \qquad \forall j>k$,

(iii) $\sup_{i\in\mathbb{N}}\bigl\|w(k,\mu_{k,i})^{*}A^{\mu_{k,i}}\bigr\|_{W^{\ell,p}(M_{k+1})}<\infty \qquad \forall \ell\in I, k\in\mathbb{N}$.

As start of the induction we give the construction for $k=1$:

Consider the sequence $(A^{\nu}|_{M_3})_{\nu\in\mathbb{N}}\subset\mathcal{A}^{1,p}(P|_{M_3})$. By assumption one finds a subsequence $(\nu_{3,i})_{i\in\mathbb{N}}$ and gauge transformations $u^{3,i}\in\mathcal{G}^{2,p}(P|_{M_3})$ such that (7.5) holds. Lemma 7.7 (ii) with $\Omega=M_2$ and $K=M_3$ then provides extensions $\tilde{u}^{3,i}\in\mathcal{G}^{2,p}_{loc}(P)$ that satisfy $\tilde{u}^{3,i}|_{M_2}=u^{3,i}|_{M_2}$. So if we set $\mu_{1,i}:=\nu_{3,i}$ and

$w(1,\mu_{1,i}) := \tilde{u}^{3,i}$, then since $M_2 \subset M_3$

$$\sup_{i\in\mathbb{N}} \left\| w(1,\mu_{1,i})^* A^{\mu_{1,i}} \right\|_{W^{\ell,p}(M_2)} = \sup_{i\in\mathbb{N}} \left\| u^{3,i\,*} A^{\nu_{3,i}} \right\|_{W^{\ell,p}(M_2)} < \infty \qquad \forall \ell \in I.$$

Now suppose for some $j \in \mathbb{N}$ that the construction has been carried out for all $k \le j$, then the construction for $k = j+1$ works as follows:

Consider the sequence $(A^{\mu_{j,i}}|_{M_{j+3}})_{i\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P|_{M_{j+3}})$. By assumption there exist a subsequence $(\nu_{j+3,i})_{i\in\mathbb{N}} \subset (\mu_{j,i})_{i\in\mathbb{N}}$ and a sequence of gauge transformations $u^{j+3,i} \in \mathcal{G}^{2,p}(P|_{M_{j+3}})$ that satisfy (7.5). Now we can use lemma 7.8 with $\Omega = M_j$, the connections $(A^{\nu_{j+3,i}})_{i\in\mathbb{N}}$, and the sequences of gauge transformations $(w(j,\nu_{j+3,i})|_{M_{j+1}})_{i\in\mathbb{N}}$ on $M'' = M_{j+1}$ and $(u^{j+3,i})_{i\in\mathbb{N}}$ on $M' = M_{j+3}$. This provides a subsequence $(\mu_{j+1,i})_{i\in\mathbb{N}} \subset (\nu_{j+3,i})_{i\in\mathbb{N}}$ and gauge transformations $w(j+1,\mu_{j+1,i}) \in \mathcal{G}^{2,p}_{loc}(P)$ such that $w(j+1,\mu_{j+1,i})|_{M_j} = w(j,\mu_{j+1,i})|_{M_j}$ and

$$\sup_{i\in\mathbb{N}} \left\| w(j+1,\mu_{j+1,i})^* A^{\mu_{j+1,i}} \right\|_{W^{\ell,p}(M_{j+3})} < \infty \qquad \forall \ell \in I. \tag{7.6}$$

Next use lemma 7.7 (with $\Omega = M_{j+2}$, $K = M_{j+3}$) to modify the gauge transformations outside of M_{j+2} and extend them to $w(j+1,\mu_{j+1,i}) \in \mathcal{G}^{2,p}_{loc}(P)$. Then (i)-(iii) are satisfied due to this and the previous construction:

(i) For all $k < j+1$

$$(\mu_{j+1,i})_{i\in\mathbb{N}} \subset (\nu_{j+3,i})_{i\in\mathbb{N}} \subset (\mu_{j,i})_{i\in\mathbb{N}} \subset (\mu_{k,i})_{i\in\mathbb{N}}.$$

(ii) For all $k < j+1$ since $M_k \subset M_j$

$$w(j+1,\mu_{j+1,i})|_{M_k} = w(j,\mu_{j+1,i})|_{M_k} = w(k,\mu_{j+1,i})|_{M_k}.$$

(iii) This follows directly from (7.6) since the gauge transformations were only modified outside of M_{j+2}, but the $W^{\ell,p}$-norms are taken on $M_{j+2} \subset M_{j+3}$.

After this construction one can now choose a diagonal subsequence that proves the proposition: Let $\nu_i := \mu_{i,i}$ and $u^i := w(i,\mu_{i,i}) \in \mathcal{G}^{2,p}_{loc}(P)$ for all $i \in \mathbb{N}$. This defines a subsequence of $\mathbb{N}$ since for all $j \in \mathbb{N}$

$$\nu_{j+1} \;=\; \mu_{j+1,j+1} \;\ge\; \mu_{j,j+1} \;>\; \mu_{j,j} \;=\; \nu_j.$$

The first inequality holds since $(\mu_{j+1,i})_{i\in\mathbb{N}}$ is a subsequence of $(\mu_{j,i})_{i\in\mathbb{N}}$. For the uniform bounds note that for all $j > k \in \mathbb{N}$

$$u^j|_{M_k} \;=\; w(j,\mu_{j,j})|_{M_k} \;=\; w(k,\mu_{j,j})|_{M_k}$$

and $(\mu_{j,i})_{i\in\mathbb{N}}$ is a subsequence of $(\mu_{k,i})_{i\in\mathbb{N}}$. Hence for every $\ell \in I$ and $k \in \mathbb{N}$

$$\begin{aligned}
&\sup_{i\in\mathbb{N}} \left\| u^{i\,*} A^{\nu_i} \right\|_{W^{\ell,p}(M_k)} \\
&= \sup\Big\{ \left\| u^{1\,*} A^{\nu_1} \right\|, \ldots, \left\| u^{k\,*} A^{\nu_k} \right\| , \sup_{j>k} \left\| w(k,\mu_{j,j})^* A^{\mu_{j,j}} \right\|_{W^{\ell,p}(M_k)} \Big\} \\
&\leq \sup\Big\{ \left\| u^{1\,*} A^{\nu_1} \right\|, \ldots, \left\| u^{k\,*} A^{\nu_k} \right\| , \sup_{i\in\mathbb{N}} \left\| w(k,\mu_{k,i})^* A^{\mu_{k,i}} \right\|_{W^{\ell,p}(M_k)} \Big\} \\
&< \infty
\end{aligned}$$

Here $\left\| u^{i\,*} A^{\nu_i} \right\|$ denotes the $W^{\ell,p}(M_k)$-norm, which is finite for all $i = 1, \ldots, k$ since $u^i|_{M_k} \in \mathcal{G}^{2,p}(P|_{M_k})$ and $A^{\nu_i}|_{M_k} \in \mathcal{A}^{1,p}(P|_{M_k})$ – see lemma A.6. □

Proof of theorem 7.5 :
The weak Uhlenbeck compactness for exhausted manifolds is a consequence of proposition 7.6 and the weak Uhlenbeck compactness on the compact submanifolds manifolds M_k :

Let a sequence $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}_{loc}(P)$ be given such that $\|F_{A^\nu}\|_{L^p(M_k)}$ is uniformly bounded for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and to check the condition of proposition 7.6 consider some subsequence of $(A^\nu|_{M_k})_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P|_{M_k})$. It satisfies the assumptions of theorem 7.1 with $M = M_k$, hence one finds a subsequence $(\nu_{k,i})_{i\in\mathbb{N}}$ and gauge transformations $u^{k,i} \in \mathcal{G}^{2,p}(P|_{M_k})$ such that $u^{k,i\,*} A^{\nu_{k,i}}$ weakly converges in $\mathcal{A}^{1,p}(P|_{M_k})$. Thus there is a uniform bound (see e.g. [Y, V.1,Thm.3])

$$\sup_{i\in\mathbb{N}} \left\| u^{k,i\,*} A^{\nu_{k,i}} \right\|_{W^{1,p}(M_k)} < \infty.$$

Now proposition 7.6 with $I = \{1\}$ provides a subsequence $(\nu_i)_{i\in\mathbb{N}}$ and a sequence of gauge transformations $u^i \in \mathcal{G}^{2,p}_{loc}(P)$ such that

$$\sup_{i\in\mathbb{N}} \left\| u^{i\,*} A^{\nu_i} \right\|_{W^{1,p}(M_k)} < \infty \qquad \forall k \in \mathbb{N}.$$

One then finds a diagonal subsequence of the connections $u^{i\,*} A^{\nu_i}$ that converges in the weak $W^{1,p}$-topology on every M_k:

This sequence of connections is uniformly $W^{1,p}$-bounded on every M_k. So by the Banach Alaoglu theorem B.4 it has a $W^{1,p}$-weakly convergent subsequence on M_1. Pick its first element for the final subsequence, then rest of the subsequence again has a subsequence that converges $W^{1,p}$-weakly on M_2. Pick its first element for the final subsequence and so on. Then after k steps one has picked k elements for a subsequence of the $u^{i\,*} A^{\nu_i}$, and all further elements will be picked from a subsequence that already converges $W^{1,p}$-weakly on M_k (and hence on M_j for all $j \leq k$). If we enumerate the resulting subsequence again by $\nu \in \mathbb{N}$ then we have constructed a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^{\nu\,*} A^\nu|_{M_k}$ weakly converges in $\mathcal{A}^{1,p}(P|_{M_k})$ for all $k \in \mathbb{N}$. □

Part III

Strong Compactness

Chapter 8

Local Slice Theorems

The key to the proof of the strong Uhlenbeck compactness theorem for Yang-Mills connections is the existence of a global relative Coulomb gauge, i.e. the local slice theorem F, that will be established in this chapter. Moreover, we prove the L^p-version, theorem F′, that provides a relative Coulomb gauge in the weak sense for connections of class L^p. The applications of this result lie beyond the scope of this book.[1] We included it since it naturally extends the usual local slice theorem for connections of class $W^{1,p}$. Moreover, its proof is actually easier and thus nicely illustrates the Newton iteration method behind the proof of theorem F.

In this chapter we consider a principal G-bundle $P \to M$ over a compact Riemannian n-manifold M, where G is a compact Lie group. Then the local slice theorems for manifolds with boundary are the following.

Theorem 8.1 (Local Slice Theorem)
Suppose that M is compact and let $1 < p \leq q < \infty$ be such that

$$p > \tfrac{n}{2} \qquad \text{and} \qquad \tfrac{1}{n} > \tfrac{1}{q} > \tfrac{1}{p} - \tfrac{1}{n}.$$

Fix a reference connection $\hat{A} \in \mathcal{A}^{1,p}(P)$ and let a constant $c_0 > 0$ be given. Then there exist constants $\delta > 0$ and C_{CG} such that the following holds. For every $A \in \mathcal{A}^{1,p}(P)$ with

$$\|A - \hat{A}\|_q \leq \delta, \qquad \|A - \hat{A}\|_{W^{1,p}} \leq c_0 \tag{8.1}$$

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(P)$ that satisfies

$$\begin{cases} \mathrm{d}_{\hat{A}}^*(u^*A - \hat{A}) = 0, \\ *(u^*A - \hat{A})|_{\partial M} = 0, \end{cases} \qquad \text{and} \qquad \begin{array}{c} \|u^*A - \hat{A}\|_q \leq C_{CG}\|A - \hat{A}\|_q, \\ \|u^*A - \hat{A}\|_{W^{1,p}} \leq C_{CG}\|A - \hat{A}\|_{W^{1,p}}. \end{array}$$

[1] Theorem F′ is used in [W] to prove that every (weakly) flat L^p-connection is gauge equivalent to a smooth connection for $p > n$.

Remark 8.2

(i) The gauge transformation u in theorem 8.1 is not necessarily unique: If $v^*A = A$ and $\hat{v}^*\hat{A} = \hat{A}$ then u can be replaced by $vu\hat{v}$. This leads to nonuniqueness if A or $\hat{A}$ has a nontrivial isotropy group.

(ii) If A and $\hat{A}$ are smooth connections then one can check that the construction in the proof of theorem 8.1 yields a smooth gauge transformation u. If moreover u is unique up to changes as in (i) then every gauge transformation that satisfies the assertion of theorem 8.1 is smooth.

(iii) When the metric in the local Coulomb gauge is varied one can still obtain uniform constants in the local slice theorem (as will be seen in the proof):

In the situation of theorem 8.1 fix in addition a metric g on M. Then there exist constants $\varepsilon, \delta > 0$ and C_{CG} such that the assertion of the theorem holds for all metrics g' with $\|g - g'\|_{W^{1,\infty}} \leq \varepsilon$. (Here we use g' in the definition of the Hodge $*$ operator but not for the Sobolev norms – these are equivalent anyway.)

(iv) Theorem 8.1 extends to the case $p > n$ and $q = \infty$. To see this, one uses the theorem for some finite $q' > n$ and notes that (8.13) in the iteration provides for some finite constants C

$$\begin{aligned}\big\|u^*A - \hat{A}\big\|_\infty &\leq \big\|A - \hat{A}\big\|_\infty + C\sum_{i=0}^{\infty}\big\|A_{i+1} - A_i\big\|_{W^{1,p}} \\ &\leq \big\|A - \hat{A}\big\|_\infty + C\big\|A - \hat{A}\big\|_{q'} \;\leq\; C\big\|A - \hat{A}\big\|_\infty .\end{aligned}$$

In order to obtain an easy expression for the weak Coulomb equation in the subsequent L^r-version of the local slice theorem we denote the dual operator of $\mathrm{d}_{\hat{A}} : W^{1,r^*}(M, \mathfrak{g}_P) \to L^{r^*}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ by

$$\mathrm{d}_{\hat{A}}{}' : L^r(M, \mathrm{T}^*M \otimes \mathfrak{g}_P) \to \big(W^{1,r^*}(M, \mathfrak{g}_P)\big)^* .$$

For $\alpha \in L^r(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ the linear form $\mathrm{d}_{\hat{A}}{}'\alpha$ acts on $W^{1,r^*}(M, \mathfrak{g}_P)$ by

$$\eta \mapsto \int_M \langle\, \alpha \,,\, \mathrm{d}_{\hat{A}}\eta \,\rangle .$$

In the general context of a vector bundle as in chapter 4 we deal with a covariant derivative $\nabla := \mathrm{d}_{\hat{A}}$ on $\mathfrak{g}_P$, its formally adjoint operator $\nabla^* = \mathrm{d}^*_{\hat{A}}$, and its dual operator $\nabla' = \mathrm{d}_{\hat{A}}{}'$. With these identifications lemma 4.1 asserts that for all $\alpha \in \Gamma(\mathrm{T}^*M \otimes \mathfrak{g}_P)$ and $\eta \in \Gamma(\mathfrak{g}_P)$

$$(\mathrm{d}_{\hat{A}}{}'\alpha)\eta = \int_M \big\langle \mathrm{d}^*_{\hat{A}}\alpha \,,\, \eta \big\rangle + \int_{\partial M} \langle\, \alpha(\nu) \,,\, \eta \,\rangle .$$

Theorem 8.3 ($\mathbf{L^r}$-Local Slice Theorem)
Suppose that M is compact, let $2 \le r < \infty$ be such that $r > n$, and fix a reference connection $\hat{A} \in \mathcal{A}^{0,r}(P)$. Then there exist constants $\delta > 0$ and C_{CG} such that the following holds. For every $A \in \mathcal{A}^{0,r}(P)$ with $\|A - \hat{A}\|_r \le \delta$ there exists a gauge transformation $u \in \mathcal{G}^{1,r}(P)$ such that

$$\mathrm{d}_{\hat{A}}{}'(u^*A - \hat{A}) = 0 \qquad \textit{and} \qquad \|u^*A - \hat{A}\|_r \le C_{CG}\|A - \hat{A}\|_r .$$

Note that the weak Coulomb equation as stated here is equivalent to the weak equation stated in theorem F′,

$$\int_M \langle u^*A - \hat{A}\,,\, \mathrm{d}_{\hat{A}}\eta \rangle = 0 \qquad \forall \eta \in \Gamma(\mathfrak{g}_P).$$

This is due to the fact that $\Gamma(\mathfrak{g}_P)$ is dense in $W^{1,r^*}(M, \mathfrak{g}_P)$.

For the rest of this chapter let r, p, and q be as assumed in above theorems. These assumptions ensure that there are Sobolev embeddings and inequalities for $W^{1,p} \hookrightarrow L^{2p}$, $W^{1,p} \hookrightarrow L^q$, $W^{1,q} \hookrightarrow \mathcal{C}^0$, and $W^{1,r} \hookrightarrow \mathcal{C}^0$, which will be used without further mention in the following.

The properties of u in theorem 8.1 (or theorem 8.3) are summarized in saying that u^*A is in (weak) Coulomb gauge relative to $\hat{A}$. If we only consider the (weak) differential equation then this is a reflexive relation, and moreover the following lemma will be useful.

Lemma 8.4

(i) Let $A^1, A^2 \in \mathcal{A}^{0,r}(P)$, then

$$\mathrm{d}_{A^1}{}'(A^2 - A^1) = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}_{A^2}{}'(A^1 - A^2) = 0.$$

(ii) Let $A^1, A^2 \in \mathcal{A}^{1,p}(P)$, then

$$\mathrm{d}^*_{A^1}(A^2 - A^1) = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}^*_{A^2}(A^1 - A^2) = 0.$$

(iii) Let $\hat{A}, A \in \mathcal{A}^{0,r}(P)$, $u \in \mathcal{G}^{1,r}(P)$, and $v = u^{-1}$, then

$$\mathrm{d}_A{}'(v^*\hat{A} - A) = 0 \qquad \Longleftrightarrow \qquad \mathrm{d}_{\hat{A}}{}'(u^*A - \hat{A}) = 0.$$

(iv) Let $\hat{A}, A \in \mathcal{A}^{1,p}(P)$, $u \in \mathcal{G}^{2,p}(P)$, and $v = u^{-1}$, then

$$\begin{cases} \mathrm{d}^*_A(v^*\hat{A} - A) = 0, \\ *(v^*\hat{A} - A)|_{\partial M} = 0, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} \mathrm{d}^*_{\hat{A}}(u^*A - \hat{A}) = 0, \\ *(u^*A - \hat{A})|_{\partial M} = 0. \end{cases}$$

Proof: For (i) and (ii) let $A := A^1 - A^2$, then for all $\eta \in W^{1,r^*}(M, \mathfrak{g}_P)$

$$\begin{aligned}
\mathrm{d}_{A^2}{}'(A^1 - A^2)\,\eta &= \int_M \big\langle\,(A^1 - A^2)\,,\,\mathrm{d}_{A_2}\eta\,\big\rangle \\
&= \int_M \big\langle\,(A^1 - A^2)\,,\,\mathrm{d}_{A_1}\eta\,\big\rangle \;-\int_M \langle\,A\,,\,[A,\eta]\,\rangle \\
&= -\mathrm{d}_{A^1}{}'(A^2 - A^1)\,\eta \;-\int_M \langle\,[A\wedge *A]\,,\,\eta\,\rangle\,. \qquad (8.2)
\end{aligned}$$

Similarly,

$$\mathrm{d}^*_{A^2}(A^1 - A^2) = -\mathrm{d}^*_{A^1}(A^2 - A^1) + *[A\wedge *A]. \qquad (8.3)$$

This proves the equivalences since the term $[\alpha\wedge *\alpha]$ vanishes in both identities. Indeed, A in locally represented by a 1-form on M with values in $\mathfrak{g}_P$. So if we calculate in local coordinates on M and remember that the inverse metric g^{ij} is symmetric but the Lie bracket is skewsymmetric then

$$*[A\wedge *A] \;=\; g^{ij}[A_i, A_j] \;=\; \sum_{i<j} g^{ij}\left([A_i, A_j] + [A_j, A_i]\right) \;=\; 0.$$

For (iii) and (iv) pick a local trivialization over some $U_\alpha \subset M$ and denote the local representatives by a subscript α. Then note that $v_\alpha \mathrm{d}u_\alpha = -\mathrm{d}v_\alpha\, u_\alpha$ due to $vu = uv = \mathbb{1}$, and hence

$$\big(v^*\hat{A} - A\big)_\alpha \;=\; u_\alpha\left(\hat{A}_\alpha + \mathrm{d}v_\alpha\, u_\alpha - v_\alpha A_\alpha u_\alpha\right) v_\alpha \;=\; u_\alpha\left(\hat{A} - u^*A\right)_\alpha v_\alpha.$$

This directly proves the equivalence of the boundary conditions. The equivalence of the first equation in (iv) is established using (8.3):

$$\begin{aligned}
&\big(\mathrm{d}^*_A(v^*\hat{A} - A)\big)_\alpha \\
&= -\mathrm{d}^*_{v^*\hat{A}}(A - v^*\hat{A}) \\
&= -\mathrm{d}^*\big(u_\alpha(u^*A - \hat{A})_\alpha\, v_\alpha\big) + *\big[(u_\alpha\hat{A}_\alpha v_\alpha + u_\alpha \mathrm{d}v_\alpha)\wedge *\, u_\alpha(u^*A - \hat{A})_\alpha\, v_\alpha\big] \\
&= -u_\alpha\, \mathrm{d}^*(u^*A - \hat{A})_\alpha\, v_\alpha + *\big[\mathrm{d}u_\alpha\, v_\alpha \wedge *\, u_\alpha(u^*A - \hat{A})_\alpha\, v_\alpha\big] \\
&\quad + *\, u_\alpha\big[\hat{A}_\alpha \wedge *(u^*A - \hat{A})_\alpha\big] v_\alpha + *\big[u_\alpha \mathrm{d}v_\alpha \wedge *\, u_\alpha(u^*A - \hat{A})_\alpha\big] v_\alpha \\
&= -u_\alpha\big(\mathrm{d}^*_{\hat{A}}(u^*A - \hat{A})\big)_\alpha\, v_\alpha.
\end{aligned}$$

For (iii) use (8.2) to obtain for all $\eta \in W^{1,r^*}(M, \mathfrak{g}_P)$

$$\begin{aligned}
\mathrm{d}_A{}'(v^*\hat{A} - A)\,\eta &= -\mathrm{d}_{v^*\hat{A}}{}'(A - v^*\hat{A})\,\eta \\
&= -\int_M \big\langle u(u^*A - \hat{A})v\,,\,\mathrm{d}_{v^*A}\,\eta\big\rangle \\
&= -\int_M \big\langle (u^*A - \hat{A})\,,\,\mathrm{d}_{\hat{A}}(v\eta u)\big\rangle \\
&= -\mathrm{d}_{\hat{A}}{}'(u^*A - \hat{A})\; v\eta u.
\end{aligned}$$

This proves the equivalence since $\eta \mapsto v\eta u$ is a bijection of $W^{1,r^*}(M, \mathfrak{g}_P)$. Indeed both u and v are of class $W^{1,r}$ by lemma A.5; they are also continuous since $r > n$. Moreover, we have used the fact that

$$\begin{aligned}\big(\mathrm{d}_{\hat{A}}(v\eta u)\big)_\alpha &= v_a \mathrm{d}\eta_\alpha\, u_\alpha + [\mathrm{d}v_\alpha\, u_\alpha\, ,\, v_\alpha \eta_\alpha u_\alpha] + [\hat{A}_\alpha\, ,\, v_\alpha \eta_\alpha u_\alpha] \\ &= v_a\big(\mathrm{d}\eta_\alpha + [u_\alpha \mathrm{d}v_\alpha\, ,\, \eta_\alpha] + [u_\alpha \hat{A}_\alpha v_\alpha\, ,\, \eta_\alpha]\big)u_\alpha \\ &= v_a\big(\mathrm{d}_{v^*\hat{A}}\eta\big)_\alpha u_\alpha\end{aligned}$$

□

The gauge transformations in the local gauge theorems will be found as the limit of some Newton iteration analogous to [CGMS, Thm.A.1]. Before we embark on the actual proofs we consider in lemma 8.5 the differential equations that will be used for the iteration, and in lemma 8.6 we prove some estimates that will be used repeatedly. But first we make some preliminary remarks.

In the case of varying metrics in remark 8.2 the Sobolev norms will all be defined with respect to one fixed metric on M. If another metric is $W^{1,\infty}$-close to this metric, then the L^r- and $W^{1,r}$-Sobolev norms for the two metrics are all equivalent since they only depend on the metric, its inverse, and first derivatives (the Christoffel symbols). Furthermore, all Sobolev norms are defined with respect to the reference connection $\hat{A}$ of the corresponding theorem. These are welldefined and still satisfy all Sobolev inequalities since they are equivalent to Sobolev norms with respect to a sufficiently close smooth reference connection $\tilde{A}$. More precicely, the L^s-norm is independent of the reference connection for all $1 \leq s \leq \infty$. In the case of theorem 8.3 the $W^{1,r}$-norms are equivalent when we choose $\|\nabla_A - \nabla_{\tilde{A}}\|_r$ sufficiently small such that for all equivariant k-forms η of class $W^{1,r}$

$$\|(\nabla_{\hat{A}} - \nabla_{\tilde{A}})\eta\|_r \;\leq\; \|\hat{A} - \tilde{A}\|_r \|\eta\|_\infty \;\leq \tfrac{1}{2}\|\eta\|_{W^{1,r},\tilde{A}}.$$

This indeed provides the equivalence (the other direction works similarly):

$$\begin{aligned}\tfrac{1}{2}\|\eta\|_{W^{1,r},\tilde{A}} &\leq \|\eta\|_r + \|\nabla_{\tilde{A}}\eta\|_r - \tfrac{1}{2}\|\eta\|_{W^{1,r},\tilde{A}} \\ &\leq \|\eta\|_r + \|\nabla_{\hat{A}}\eta\|_r \\ &\leq 2\|\eta\|_{W^{1,r}}.\end{aligned}$$

The W^{1,r^*}-norms are also equivalent but by a different estimate: For sufficiently small $\|\hat{A} - \tilde{A}\|_r$ and all equivariant k-forms η of class W^{1,r^*}

$$\|(\nabla_{\hat{A}} - \nabla_{\tilde{A}})\eta\|_{r^*} \;\leq\; \|\hat{A} - \tilde{A}\|_r \|\eta\|_s \;\leq\; \tfrac{1}{2}\|\eta\|_{W^{1,r^*}}.$$

Here $\frac{1}{s} = \frac{1}{r^*} - \frac{1}{r} = 1 - \frac{2}{r}$ defines $s \in [1, \infty]$ since $r \geq 2$, and the Sobolev embedding $W^{1,r^*} \hookrightarrow L^s$ holds due to $r > n$:

$$\tfrac{1}{n} - \tfrac{1}{r^*} \;=\; \tfrac{1}{n} - 1 + \tfrac{1}{r} \;>\; -1 + \tfrac{2}{r} \;=\; -\tfrac{1}{s}.$$

Now for all $\alpha \in L^r(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ and $\eta \in W^{1,r^*}(M, \mathfrak{g}_P)$ one can use the fact that $\left|\int_M \langle \alpha , \mathrm{d}_{\hat{A}}\eta \rangle\right| \leq \|\alpha\|_r \|\mathrm{d}_{\hat{A}}\eta\|_{r^*} \leq \|\alpha\|_r \|\eta\|_{W^{1,r^*}}$ to obtain

$$\|\mathrm{d}_{\hat{A}}{}'\alpha\|_{(W^{1,r^*})^*} \;=\; \sup\left\{ \frac{\left|\int_M \langle \alpha , \mathrm{d}_{\hat{A}}\eta \rangle\right|}{\|\eta\|_{W^{1,r^*}}} \,\middle|\, \eta \in W^{1,r^*}(M, \mathfrak{g}_P) \right\} \;\leq\; \|\alpha\|_r. \quad (8.4)$$

In the case of theorem 8.1 the $W^{1,s}$-norms are equivalent as long as $\frac{1}{s} \geq \frac{1}{p} - \frac{1}{n}$ (especially for $s = p$ and $s = q$), and also the $W^{2,p}$-norms are equivalent.

To see this let $\frac{1}{s} = \frac{1}{p'} + \frac{1}{s'}$ such that there are Sobolev embeddings $W^{1,p} \hookrightarrow L^{p'}$ and $W^{1,s} \hookrightarrow L^{s'}$. (In case $s > n$ one can choose $s' = \infty$ and $p' = s$. In case $s \leq n$ one finds a $p' > n$ since $\frac{1}{n} > \frac{1}{p} - \frac{1}{n}$, and then automatially $\frac{1}{s'} = \frac{1}{s} - \frac{1}{p'} \geq \frac{1}{s} - \frac{1}{n}$ ensures the other Sobolev embedding.) Then there is a constant C such that for all smooth equivariant k-forms η

$$\begin{aligned} \|(\nabla_{\hat{A}} - \nabla_{\tilde{A}})\eta\|_s &= \big\|[(\hat{A} - \tilde{A}) \wedge \eta]\big\|_s \\ &\leq 2\|\hat{A} - \tilde{A}\|_{p'} \|\eta\|_{s'} \\ &\leq C\|\hat{A} - \tilde{A}\|_{W^{1,p},\tilde{A}} \|\eta\|_{W^{1,s},\tilde{A}}. \end{aligned} \quad (8.5)$$

Here the fact that the Sobolev norm is defined with respect to $\tilde{A}$ is indicated by an additional subscript. If we choose $\|\hat{A} - \tilde{A}\|_{W^{1,p},\tilde{A}}$ sufficiently small then this shows the equivalence.

For the $W^{2,p}$-norms moreover use lemma B.3 (with $r = p$, $s = q$) to obtain for all smooth equivariant k-forms η and some constants C, C'

$$\begin{aligned} &\|(\nabla_{\hat{A}}^2 - \nabla_{\tilde{A}}^2)\eta\|_p \\ &\leq \big\|\nabla_{\tilde{A}}[(\hat{A} - \tilde{A}) \wedge \eta]\big\|_p + \big\|[(\hat{A} - \tilde{A}) \wedge \nabla_{\tilde{A}}\eta]\big\|_p + \big\|\big[(\hat{A} - \tilde{A}) \wedge [(\hat{A} - \tilde{A}) \wedge \eta]\big]\big\|_p \\ &\leq C'\|\hat{A} - \tilde{A}\|_{W^{1,p},\tilde{A}} \|\eta\|_{W^{1,q},\tilde{A}} + \|\hat{A} - \tilde{A}\|_{2p} \|\nabla_{\tilde{A}}\eta\|_{2p} + \|\hat{A} - \tilde{A}\|_{2p}^2 \|\eta\|_\infty \\ &\leq C\|\hat{A} - \tilde{A}\|_{W^{1,p},\tilde{A}} \big(1 + \|\hat{A} - \tilde{A}\|_{W^{1,p},\tilde{A}}\big) \|\eta\|_{W^{2,p},\tilde{A}}. \end{aligned}$$

Moreover, there exists a constant C such that for all $\alpha \in W^{1,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$

$$\|\mathrm{d}_{\hat{A}}^*\alpha\|_p + \|*\alpha|_{\partial M}\|_{W_\partial^{1,p}} \leq C\|\alpha\|_{W^{1,p}}. \quad (8.6)$$

This also holds with a uniform constant C when the metric in the definition of $\mathrm{d}_{\hat{A}}^*$ and $*$ is varied within a $W^{1,\infty}$-small neighbourhood of a given metric.

Indeed, if $\hat{A}$ is replaced by a smooth connection $\tilde{A}$ then for the first term this inequality is obvious from the Sobolev norm in bundle charts. The difference $\mathrm{d}_{\hat{A}}^*\alpha - \mathrm{d}_{\tilde{A}}^*\alpha = *\big[(\tilde{A} - \hat{A}) \wedge *\alpha\big]$ is estimated as in (8.5). For the second term extend the outward unit normal vector ν to a smooth vector field $\tilde{\nu}$ on M, then one has $*\alpha|_{\partial M} = \alpha(\nu)\,\mathrm{dvol}_{\partial M}$ and $\|\alpha(\nu)\|_{W_\partial^{1,p}} \leq \|\alpha(\tilde{\nu})\|_{W^{1,p}} \leq C\|\alpha\|_{W^{1,p}}$ by lemma 5.6 (i). When the metric varies note that $\mathrm{d}_{\hat{A}}^*$ depends on the metric only up to its first derivatives. Moreover, an extension $\tilde{\nu}_{g'}$ of the unit normal with respect to a metric g' can be chosen $W^{1,\infty}$-close to the extension $\tilde{\nu}$ for a fixed metric g (in local coordinates $\tilde{\nu}^i_{g'} = g'^{ij} g_{jk} \tilde{\nu}^k$).

Lemma 8.5

(i) Let $\hat{A} \in \mathcal{A}^{0,r}(P)$. Then there exists a constant C_1 such that for every $\alpha \in L^r(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ there exists a solution $\xi \in W^{1,r}(M, \mathfrak{g}_P)$ of

$$\mathrm{d}_{\hat{A}}{}'\mathrm{d}_{\hat{A}}\xi = \mathrm{d}_{\hat{A}}{}'\alpha \qquad \textit{with} \qquad \|\xi\|_{W^{1,r}} \le C_1 \|\mathrm{d}_{\hat{A}}{}'\alpha\|_{(W^{1,r^*})^*}.$$

(ii) Let $\hat{A} \in \mathcal{A}^{1,p}(P)$. Then there exists a constant C_1 such that for every $\alpha \in W^{1,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ there exists a solution $\xi \in W^{2,p}(M, \mathfrak{g}_P)$ of

$$\begin{cases} \mathrm{d}^*_{\hat{A}}\mathrm{d}_{\hat{A}}\xi = \mathrm{d}^*_{\hat{A}}\alpha, \\ *\mathrm{d}_{\hat{A}}\xi|_{\partial M} = *\alpha|_{\partial M} \end{cases} \quad \textit{with} \quad \begin{array}{l} \|\xi\|_{W^{2,p}} \le C_1\big(\|\mathrm{d}^*_{\hat{A}}\alpha\|_p + \|*\alpha|_{\partial M}\|_{W^{1,p}_\partial}\big), \\ \|\xi\|_{W^{1,q}} \le C_1\|\alpha\|_q. \end{array}$$

(iii) Let $\hat{A} \in \mathcal{A}^{1,p}(P)$ and fix a metric g on M. Then there exist constants $\varepsilon > 0$ and C_1 such that the assertion of (ii) holds for all metrics g' with $\|g - g'\|_{W^{1,\infty}} \le \varepsilon$. (The metric g' replaces g in the boundary value problem but not in the Sobolev norms – these are equivalent anyway.)

Proof: These are theorem 4.3 and 4.4 and remark 4.5 for the vector bundle $E = \mathfrak{g}_P$ with the covariant derivative $\nabla = \mathrm{d}_{\hat{A}}$. The inner product on the fibres is induced by the metric of $\mathfrak{g}$ from theorem A.2. The connection potentials of ∇ are the local representatives of the connection 1-form $\hat{A}$, hence they are L^r-regular in (i) and $W^{1,p}$-regular in (ii). Here an element $\eta \in \mathfrak{g}$ of the Lie algebra is understood as endomorphism of $\mathfrak{g}$ via the adjoint action $\xi \mapsto [\eta, \xi]$. By (A.5) the adjoint of this endomorphism is $\xi \mapsto -[\eta, \xi]$. So the formally adjoint operator of ∇ is indeed $\nabla^* = \mathrm{d}^*_{\hat{A}}$, see (A.9), and the dual operator of ∇ is $\nabla' = \mathrm{d}_{\hat{A}}{}'$. The boundary condition is equivalent to $\mathrm{d}_{\hat{A}}\xi(\nu) = \alpha(\nu)$ on ∂M, see lemma 5.6 (i). Similarly, the boundary value $*\alpha|_{\partial M} = \alpha(\nu)\,\mathrm{dvol}_{\partial M}$ is identified with $\alpha(\nu)$ to obtain the estimate. □

In the next lemma we do not need to consider different metrics on M since the metric for the Sobolev norms is fixed anyway and the terms $\exp(\xi)^*A - A$ and $\mathrm{d}_A\xi$ are independent of the metric.

Lemma 8.6

(i) Let $\hat{A} \in \mathcal{A}^{0,r}(P)$, then for every constant c_2 there exists a constant C_2 such that the following holds: For all $A \in \mathcal{A}^{0,r}(P)$ and $\xi \in W^{1,r}(M, \mathfrak{g}_P)$ with $\|\xi\|_{W^{1,r}} \le c_2$

$$\big\|\exp(\xi)^*A - A - \mathrm{d}_A\xi\big\|_r \le C_2\big(1 + \|A - \hat{A}\|_r\big)\|\xi\|^2_{W^{1,r}}$$
$$\big\|\exp(\xi)^*A - A\big\|_r \le C_2\big(1 + \|A - \hat{A}\|_r\big)\|\xi\|_{W^{1,r}}.$$

(ii) *Let $\hat{A} \in \mathcal{A}^{1,p}(P)$, then for every constant c_2 there exists a constant C_2 such that the following holds: For all $A \in \mathcal{A}^{1,p}(P)$ and $\xi \in W^{2,p}(M, \mathfrak{g}_P)$ with $\|\xi\|_{W^{2,p}} \le c_2$*

$$\begin{aligned}\big\|\exp(\xi)^*A - A - \mathrm{d}_A\xi\big\|_{W^{1,p}} &\le C_2\big(1 + \|A - \hat{A}\|_{W^{1,p}}\big)\|\xi\|_{W^{1,q}}\|\xi\|_{W^{2,p}},\\ \big\|\exp(\xi)^*A - A\big\|_{W^{1,p}} &\le C_2\big(1 + \|A - \hat{A}\|_{W^{1,p}}\big)\|\xi\|_{W^{2,p}}.\end{aligned}$$

Proof: For both (i) and (ii) consider the function $\alpha(t) := \exp(t\xi)^*A - A$. This is a $\mathcal{C}^1$-map from $[0,1]$ to $L^r(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ or $W^{1,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ and it satisfies $\alpha(0) = 0$ and

$$\begin{aligned}\dot\alpha(t) &= \tfrac{\partial}{\partial s}\big|_{s=0}\big(e^{-(s+t)\xi}Ae^{(s+t)\xi} + e^{-(s+t)\xi}\mathrm{d}e^{(s+t)\xi} - A\big)\\ &= \tfrac{\partial}{\partial s}\big|_{s=0}\, e^{-s\xi}\big(e^{-t\xi}Ae^{t\xi} + e^{-t\xi}\mathrm{d}e^{t\xi}\big)e^{s\xi} + \tfrac{\partial}{\partial s}\big|_{s=0}\, e^{-s\xi}\mathrm{d}e^{s\xi}\\ &= \big[-\xi, e^{-t\xi}Ae^{t\xi} + e^{-t\xi}\mathrm{d}e^{t\xi}\big] + \mathrm{d}\xi\\ &= \mathrm{d}_A\xi - \mathrm{ad}_\xi(\alpha(t)).\end{aligned}$$

This calculation works with the local representations of A and ξ, that are denoted by the same letters. With this local representative ξ one has $e^{(t+s)\xi} = e^{t\xi}e^{s\xi}$ – see remark A.3 (iii). Moreover, let L_g denote the left multiplication by $g \in G$, then

$$\begin{aligned}\big(\tfrac{\partial}{\partial s}\big|_{s=0}\, e^{-s\xi}\mathrm{d}e^{s\xi}\big)(p) &= \tfrac{\partial}{\partial s}\big|_{s=0}\big(L_{\exp(-s\xi(p))}\circ\mathrm{d}_{s\xi(p)}\exp\circ s\cdot\mathrm{d}_p\xi\big)\\ &= L_{\mathbb{1}}\circ\mathrm{d}_0\exp\circ\tfrac{\partial}{\partial s}\big|_{s=0}(s\cdot\mathrm{d}_p\xi) \;=\; \mathrm{d}_p\xi.\end{aligned}$$

This is a composition of three linear operators depending on s, the last of which, $s\cdot\mathrm{d}_p\xi$, vanishes at $s = 0$, hence one does not need to calculate the other differentials explicitly.

The continuity of α is due to the continuity of the gauge action, lemma A.6; the continuity of $\dot\alpha$ is seen from above differential equation. The solution of this initial value problem for α is unique, [2]

$$\alpha(t) = t\cdot\mathrm{d}_A\xi + \sum_{k=1}^{\infty}\tfrac{(-t)^k}{(k+1)!}\mathrm{ad}_\xi^k(\mathrm{d}_A\xi).$$

For $t = 1$ this provides

$$\exp(\xi)^*A - A - \mathrm{d}_A\xi = \sum_{k=1}^{\infty}\tfrac{(-1)^k}{(k+1)!}\mathrm{ad}_\xi^k(\mathrm{d}_A\xi).$$

In the following we denote all finite constants by C. The Sobolev inequality for $W^{1,r} \hookrightarrow \mathcal{C}^0$ in case (i) and for $W^{2,p} \hookrightarrow \mathcal{C}^0$ in case (ii) now provides a constant C

[2] The difference of any two solutions, $\beta \in \mathcal{C}^1\big([0,1], W^{1,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)\big)$, satisfies $\beta(0) = 0$ and $\dot\beta(t) = F(\beta(t))$, where $F : \beta \mapsto [\beta, \xi]$ is a bounded linear operator and hence Lipschitz continuous.

such that all relevant ξ satisfy $\|\xi\|_\infty \leq C$. Thus one can use (A.6) to obtain

$$\begin{aligned}\big|\exp(\xi)^*A - A - \mathrm{d}_A\xi\big| &\leq \sum_{k=1}^{\infty} \tfrac{C^{k-1}}{(k+1)!} \cdot |\xi| \cdot |\mathrm{d}_A\xi| \\ &= \tfrac{e^C - 1 - C}{C^2} \cdot |\xi| \cdot |\mathrm{d}_A\xi| \\ &\leq C\big(|\xi| \cdot |\nabla_{\hat{A}}\xi| + |A - \hat{A}| \cdot |\xi|^2\big)\end{aligned}$$

This suffices to prove the first estimate in (i) :

$$\begin{aligned}\big\|\exp(\xi)^*A - A - \mathrm{d}_A\xi\big\|_r &\leq C\big(\|\xi\|_\infty \|\xi\|_{W^{1,r}} + \|A - \hat{A}\|_r \|\xi\|_\infty^2\big) \\ &\leq C\big(1 + \|A - \hat{A}\|_r\big)\|\xi\|_{W^{1,r}}^2 .\end{aligned}$$

The second estimate in (i) follows from the first one since $\|\xi\|_{W^{1,r}} \leq c_2$ and

$$\begin{aligned}\|\mathrm{d}_A\xi\|_r \;&\leq\; \|\nabla_{\hat{A}}\xi\|_r + \big\|[A - \hat{A}, \xi]\big\|_r \;\leq\; \|\xi\|_{W^{1,r}} + \|A - \hat{A}\|_r \|\xi\|_\infty \\ &\leq\; C\big(1 + \|A - \hat{A}\|_r\big)\|\xi\|_{W^{1,r}} .\end{aligned}$$

For (ii) one moreover has

$$\begin{aligned}&\big|\nabla_{\hat{A}}\big(\exp(\xi)^*A - A - \mathrm{d}_A\xi\big)\big| \\ &\leq \sum_{k=1}^{\infty} \tfrac{C^{k-1}}{(k+1)!}\big(|\nabla_{\hat{A}}\xi| \cdot |\mathrm{d}_A\xi| + |\xi| \cdot |\nabla_{\hat{A}}\mathrm{d}_A\xi|\big) \\ &\leq \tfrac{e^C - 1 - C}{C^2}\big(|\nabla_{\hat{A}}\xi|^2 + |\nabla_{\hat{A}}\xi| \cdot |A - \hat{A}| \cdot |\xi| + |\xi| \cdot |\nabla_{\hat{A}}^2\xi| + |\xi| \cdot |\nabla_{\hat{A}}[A - \hat{A}, \xi]|\big) \\ &\leq C\big(|\nabla_{\hat{A}}\xi|^2 + |A - \hat{A}| \cdot |\xi| \cdot |\nabla_{\hat{A}}\xi| + |\xi| \cdot |\nabla_{\hat{A}}^2\xi| + |\nabla_{\hat{A}}(A - \hat{A})| \cdot |\xi|^2\big).\end{aligned}$$

This proves the first estimate: Let $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$, then $\frac{1}{s} > \frac{1}{p} - \frac{1}{n}$ ensures the Sobolev inequality for $W^{2,p} \hookrightarrow W^{1,s}$ (and the equivalence of the $W^{1,s}$-norm with respect to $\hat{A}$ to a Sobolev norm for a smooth reference connection), so we obtain

$$\begin{aligned}&\big\|\exp(\xi)^*A - A - \mathrm{d}_A\xi\big\|_{W^{1,p}} \\ &\leq \big\|\exp(\xi)^*A - A - \mathrm{d}_A\xi\big\|_p + \big\|\nabla_{\hat{A}}\big(\exp(\xi)^*A - A - \mathrm{d}_A\xi\big)\big\|_p \\ &\leq C\big(\; \|\xi\|_{W^{1,q}}\|\xi\|_{W^{1,s}} + \|A - \hat{A}\|_{2p}\|\xi\|_\infty\|\xi\|_{W^{1,2p}} + \|\xi\|_\infty\|\xi\|_{W^{2,p}} \\ &\qquad + \|A - \hat{A}\|_{W^{1,p}}\|\xi\|_\infty^2 \;\big) \\ &\leq C\big(1 + \|A - \hat{A}\|_{W^{1,p}}\big)\|\xi\|_{W^{1,q}}\|\xi\|_{W^{2,p}} .\end{aligned}$$

The second estimate in (ii) follows from the first one. One only has to note that $\|\xi\|_{W^{1,q}} \leq C\|\xi\|_{W^{2,p}} \leq Cc_2$ and use lemma B.3 with $r = p$, $s = q$ in

$$\begin{aligned}\|\mathrm{d}_A\xi\|_{W^{1,p}} &\leq \|\nabla_{\hat{A}}\xi\|_{W^{1,p}} + \big\|[A - \hat{A}, \xi]\big\|_{W^{1,p}} \\ &\leq \|\xi\|_{W^{2,p}} + C\|A - \hat{A}\|_{W^{1,p}}\|\xi\|_{W^{1,q}} \\ &\leq C\big(1 + \|A - \hat{A}\|_{W^{1,p}}\big)\|\xi\|_{W^{2,p}} .\end{aligned}$$

□

Finally, we proceed to prove the two local slice theorems 8.3 and 8.1. The first could actually be deduced from the implicit function theorem and lemma 8.6. Theorem 8.1 goes beyond the reach of the implicit function theorem since the relevant $W^{1,p}$-norm is only bounded, not small. So the latter theorem is proven by a Newton iteration method, and we also exemplify this method in the easier case of theorem 8.3.

Proof of theorem 8.3 :
Fix a reference connection $\hat{A} \in \mathcal{A}^{0,r}(P)$ and consider a connection $A \in \mathcal{A}^{0,r}(P)$ that satisfies $\|A - \hat{A}\|_r \leq \delta$ for some $1 \geq \delta > 0$ (this will be specified later) .

The idea of the proof is to use Newtons iteration method to solve the equation $\mathrm{d}_{\hat{A}}{}'(u^*A - \hat{A}) = 0$ for u. This leads to the definition of connections A_i and gauge transformations $u_i = \exp(\xi_1)\ldots\exp(\xi_i)$ such that $u_i^*A = A_i$. We will first show that A_i converges to a connection A_∞ that is in relative Coulomb gauge with respect to $\hat{A}$, i.e. $\mathrm{d}_{\hat{A}}{}'(A_\infty - \hat{A}) = 0$, and then see that in fact $A_\infty = u^*A$ for some gauge transformation u.

So define sequences of gauge transformations $\exp(\xi_i) \in \mathcal{G}^{1,r}(P)$ and connections $A_i \in \mathcal{A}^{0,r}(P)$ by the following Newton iteration: $A_0 := A$ and $A_{i+1} := \exp(\xi_i)^*A_i$, where $\xi_i \in W^{1,r}(M, \mathfrak{g}_P)$ is the solution of

$$\mathrm{d}_{\hat{A}}{}'\mathrm{d}_{\hat{A}}\xi_i = \mathrm{d}_{\hat{A}}{}'(\hat{A} - A_i)$$

provided by lemma 8.5 (i). This solution moreover satisfies – using (8.4) –

$$\|\xi_i\|_{W^{1,r}} \;\leq\; C_1\big\|\mathrm{d}_{\hat{A}}{}'(A_i - \hat{A})\big\|_{(W^{1,r^*})^*} \;\leq\; C_1\|A_i - \hat{A}\|_r. \tag{8.7}$$

We claim that for sufficiently small $\delta > 0$ this sequence satisfies for all $i \in \mathbb{N}_0$

$$\big\|\mathrm{d}_{\hat{A}}{}'(A_i - \hat{A})\big\|_{(W^{1,r^*})^*} \leq 2^{-i}\|A - \hat{A}\|_r, \tag{8.8}$$

and moreover there exists a constant C_I such that for all $i \geq 1$

$$\|A_i - A_{i-1}\|_r \leq 2^{-i}C_I\|A - \hat{A}\|_r. \tag{8.9}$$

In the following let C_2 be the constant from lemma 8.6 (i) for $c_2 = C_1$. We will prove (8.8) and (8.9) with $C_I = 4C_1C_2$ by induction, making a sufficiently small choice of $\delta > 0$ (independently of i) in the induction step.

As start of the induction (8.8) holds for $i = 0$ due to (8.4) :

$$\big\|\mathrm{d}_{\hat{A}}{}'(A_0 - \hat{A})\big\|_{(W^{1,r^*})^*} \;\leq\; \|A_0 - \hat{A}\|_r \;=\; \|A - \hat{A}\|_r.$$

Now assume that (8.8) and (8.9) are established for all $i \leq j$ with some $j \in \mathbb{N}_0$ (in case $j = 0$ this only means (8.8)). The induction step is then to prove (8.8) and (8.9) for $i = j+1$: First use (8.9) for all $i \leq j$ to obtain

$$\begin{aligned}\|A_j - \hat{A}\|_r \;&\leq\; \|A_0 - \hat{A}\|_r + \sum_{i=1}^{j}\|A_i - A_{i-1}\|_r \;\leq\; \Big(1 + C_I\sum_{i=1}^{j}2^{-i}\Big)\|A - \hat{A}\|_r \\ &\leq\; C_{CG}\|A - \hat{A}\|_r \;\leq\; 1.\end{aligned} \tag{8.10}$$

Here the constant $C_{CG} := 1 + C_I$ is independent of j, and we choose $\delta \leq C_{CG}^{-1}$. Next, (8.7) and (8.10) provide the bound for lemma 8.6(i) that allows to use the estimates with the constant C_2 fixed above,

$$\|\xi_j\|_{W^{1,r}} \;\leq\; C_1\|A_j - \hat{A}\|_r \;\leq\; C_1 \;=\; c_2.$$

Now use $\mathrm{d}_{\hat{A}}{}'\hat{A} = \mathrm{d}_{\hat{A}}{}'\big(\mathrm{d}_{\hat{A}}\xi_j + A_j\big)$ to rewrite

$$\mathrm{d}_{\hat{A}}{}'(A_{j+1} - \hat{A}) = \mathrm{d}_{\hat{A}}{}'\big(\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big) + \mathrm{d}_{\hat{A}}{}'\big[A_j - \hat{A}, \xi_j\big].$$

The first term on the right hand side can be estimated by lemma 8.6 (i) and with the help of (8.4), (8.7), and (8.10) :

$$\begin{aligned}
\big\|\mathrm{d}_{\hat{A}}{}'\big(\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big)\big\|_{(W^{1,r^*})^*} \hspace{-6em}&\\
&\leq \big\|\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big\|_r \\
&\leq C_2\big(1 + \|A_j - \hat{A}\|_r\big)\|\xi_j\|_{W^{1,r}}^2 \\
&\leq 2C_1^2 C_2\|A_j - \hat{A}\|_r\big\|\mathrm{d}_{\hat{A}}{}'(A_j - \hat{A})\big\|_{(W^{1,r^*})^*}.
\end{aligned}$$

For the second term use (8.4), (8.7), and (8.10) to obtain

$$\begin{aligned}
\big\|\mathrm{d}_{\hat{A}}{}'[A_j - \hat{A}, \xi_j]\big\|_{(W^{1,r^*})^*} &\leq \big\|[A_j - \hat{A}, \xi_j]\big\|_r \\
&\leq \|A_j - \hat{A}\|_r\|\xi_j\|_\infty \\
&\leq C\|A_j - \hat{A}\|_r\|\xi_j\|_{W^{1,r}} \\
&\leq CC_1\|A_j - \hat{A}\|_r\big\|\mathrm{d}_{\hat{A}}{}'(A_j - \hat{A})\big\|_{(W^{1,r^*})^*}.
\end{aligned}$$

Putting this back together proves (8.8) for $i = j+1$: Use (8.8) for $i = j$ and choose $\delta \leq \frac{1}{2}((2C_1C_2 + C)C_1C_{CG})^{-1}$ (which is independent of j), then

$$\begin{aligned}
\|\mathrm{d}_{\hat{A}}{}'(A_{j+1} - \hat{A})\|_{(W^{1,r^*})^*} &\leq (2C_1^2C_2 + CC_1)\|A_j - \hat{A}\|_r\big\|\mathrm{d}_{\hat{A}}{}'(A_j - \hat{A})\big\|_{(W^{1,r^*})^*} \\
&\leq (2C_1C_2 + C)C_1C_{CG}\delta\, 2^{-j}\|A - \hat{A}\|_r \\
&\leq 2^{-j-1}\|A - \hat{A}\|_r.
\end{aligned}$$

Furthermore, (8.9) for $i = j+1$ also follows from (8.8) for $i = j$ with the help of lemma 8.6 (i), (8.10), and (8.7):

$$\begin{aligned}
\|A_{j+1} - A_j\|_r &= \|\exp(\xi_j)^* A_j - A_j\|_r \\
&\leq C_2\big(1 + \|A_j - \hat{A}\|_r\big)\|\xi_j\|_{W^{1,r}} \\
&\leq 2C_1C_2\|\mathrm{d}_{\hat{A}}{}'(A_j - \hat{A})\|_{(W^{1,r^*})^*} \\
&\leq 2^{-j+1}C_1C_2\|A - \hat{A}\|_r.
\end{aligned}$$

Since $C_I = 4C_1C_2$ this finishes the induction step.

So we have shown that the Newton sequence A_i of connections satisfies (8.8) and (8.9). The latter shows that the A_i form a L^r-Cauchy sequence. Indeed, for all $k > j \geq 1$

$$\|A_k - A_j\|_r \;\leq\; \sum_{i=j+1}^{k} \|A_i - A_{i-1}\|_r \;\leq\; \sum_{i=j+1}^{k} 2^{-i} C_I \|A - \hat{A}\|_r \;\leq\; 2^{-j}\delta C_I .$$

Since $\mathcal{A}^{0,r}(P)$ is a Banach space this implies that the A_i converge in the L^r-norm to some $A_\infty \in \mathcal{A}^{0,r}(P)$. The inequality (8.10) is preserved under this limit,

$$\|A_\infty - \hat{A}\|_r \leq C_{CG} \|A - \hat{A}\|_r .$$

Moreover, ${\rm d}_{\hat{A}}{}'$ is a continuous map from $L^r(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ to $(W^{1,r^*}(M, \mathfrak{g}_P))^*$ (see (ii) in the proof of theorem 4.3), so (8.8) shows that

$$\mathrm{d}_{\hat{A}}{}'(A_\infty - \hat{A}) \;=\; \lim_{i\to\infty} \mathrm{d}_{\hat{A}}{}'(A_i - \hat{A}) \;=\; 0.$$

It remains to show that $A_\infty = u^*A$ for some $u \in \mathcal{G}^{1,r}(P)$. For that purpose consider the sequence $u_i = \exp(\xi_1)\ldots\exp(\xi_i)$. By lemma A.5 it lies in $\mathcal{G}^{1,r}(P)$, and it moreover satisfies $u_i^*A = A_i$. Now lemma A.8 applies (with $k = 1$, $p = r$, and $s = r$) since the A_i converge in the L^r-norm and A is uniformly L^r-bounded anyway. Thus there exists a subsequence of the u_i that converges in the $\mathcal{C}^0$-norm to some $u \in \mathcal{G}^{1,r}(P)$. For the same subsequence (again labelled by i) $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in the L^r-norm. Now we obtain $u^*A = A_\infty$ since this is the unique L^r-limit of the sequence

$$u_i^{-1}Au_i + u_i^{-1}\mathrm{d}u_i \;=\; u_i^*A \;=\; A_i .$$

Thus u is the required gauge transformation that puts A in relative Coulomb gauge with respect to $\hat{A}$. □

Proof of theorem 8.1 :
Fix a connection $\hat{A} \in \mathcal{A}^{1,p}(P)$ and a constant $c_0 > 0$ and consider a connection $A \in \mathcal{A}^{1,p}(P)$ that satisfies (8.1) for some $\delta > 0$. Again the idea of the proof is to use Newtons iteration method to solve the boundary value problem for u. One defines connections A_i and gauge transformations $u_i = \exp(\xi_1)\ldots\exp(\xi_i)$ such that $u_i^*A = A_i$ and A_i converges to a connection A_∞ that is in relative Coulomb gauge with respect to $\hat{A}$. Then one proves that in fact $A_\infty = u^*A$ for some gauge transformation u.

In the case of varying metrics in remark 8.2 one chooses the $W^{1,\infty}$-neighbourhood of the given metric g as in lemma 8.5 (iii). Moreover, choose this neighbourhood, that is $\varepsilon > 0$, sufficiently small such that (8.6) holds with a uniform constant for all metrics g' that satisfy $\|g - g'\|_{W^{1,\infty}} \leq \varepsilon$. Then all constants in the following will be independent of the metric g' that is used in the boundary value problem. The constants in Sobolev inequalities are also independent of g' since they are defined with respect to g. That way the local slice theorem is proven with uniform constants for all metrics in the $W^{1,\infty}$-neighbourhood of g.

So we construct the sequences of gauge transformations $\exp(\xi_i) \in \mathcal{G}^{2,p}(P)$ and connections $A_i \in \mathcal{A}^{1,p}(P)$ by the following Newton iteration: $A_0 := A$ and $A_{i+1} := \exp(\xi_i)^* A_i$, where $\xi_i \in W^{2,p}(M, \mathfrak{g}_P)$ is provided by lemma 8.5 (ii). It is the solution of

$$\begin{cases} \mathrm{d}_{\hat{A}}^* \mathrm{d}_{\hat{A}} \xi_i = \mathrm{d}_{\hat{A}}^* (\hat{A} - A_i), \\ * \mathrm{d}_{\hat{A}} \xi_i |_{\partial M} = *(\hat{A} - A_i)|_{\partial M}, \end{cases}$$

with

$$\begin{aligned} \|\xi_i\|_{W^{2,p}} &\leq C_1 \big(\|\mathrm{d}_{\hat{A}}^* (A_i - \hat{A})\|_p + \| * (A_i - \hat{A})|_{\partial M} \|_{W^{1,p}_\partial} \big), \\ \|\xi_i\|_{W^{1,q}} &\leq C_1 \|A_i - \hat{A}\|_q. \end{aligned} \tag{8.11}$$

We claim that for sufficiently small $\delta > 0$ there exist constants C_I, C_{II} such that this sequence satisfies for all $i \in \mathbb{N}$

$$\|\mathrm{d}_{\hat{A}}^* (A_i - \hat{A})\|_p + \| * (A_i - \hat{A})|_{\partial M} \|_{W^{1,p}_\partial} \leq 2^{-i} C_I \|A - \hat{A}\|_q, \tag{8.12}$$

and moreover for all $i \geq 2$

$$\|A_i - A_{i-1}\|_{W^{1,p}} \leq 2^{-i} C_{II} \|A - \hat{A}\|_q. \tag{8.13}$$

In the following let C_2 be the constant from lemma 8.6 (ii) for $c_2 = C_1 C_3 c_0$, where C_3 denotes the constant from (8.6). The constants C_I and C_{II} will be determined by c_0, C_1, C_2, C_3, and some Sobolev constants C. The induction step for (8.12) and (8.13) will require a sufficiently small choice of $\delta > 0$, depending on C_I and C_{II}. This is the same proceeding as in the proof of theorem 8.3 – we first fix C_I and C_{II} and then determine a suitable $\delta > 0$, just that we do not give the more complicated formulae here.

Even before starting the induction we choose $\delta \leq c_0 C_{II}^{-1}$. Now assume that (8.13) holds for all $i \leq j \in \mathbb{N}$, then this implies

$$\begin{aligned} \|A_j - \hat{A}\|_{W^{1,p}} &\leq \|A_0 - \hat{A}\|_{W^{1,p}} + \sum_{i=1}^{j} \|A_i - A_{i-1}\|_{W^{1,p}} \\ &\leq \|A - \hat{A}\|_{W^{1,p}} + \sum_{i=1}^{j} 2^{-i} C_{II} \|A - \hat{A}\|_q \\ &\leq \|A - \hat{A}\|_{W^{1,p}} + C_{II} \|A - \hat{A}\|_q \\ &\leq c_0 + C_{II} \delta \ \leq \ 2 c_0. \end{aligned} \tag{8.14}$$

Similarly, this implies with a Sobolev constant C

$$\begin{aligned}\|A_j - \hat{A}\|_q &\leq \|A_0 - \hat{A}\|_q + \sum_{i=1}^{j} C\|A_i - A_{i-1}\|_{W^{1,p}} \\ &\leq \|A - \hat{A}\|_q + \sum_{i=1}^{j} 2^{-i} C C_{I\!I} \|A - \hat{A}\|_q \\ &\leq (1 + CC_{I\!I})\|A - \hat{A}\|_q \;\leq\; (1 + CC_{I\!I})\delta. \end{aligned} \tag{8.15}$$

Note that both (8.14) and (8.15) are true for $j = 0$ by assumption. That can be used as start of the induction. Then for the induction step suppose that (8.12) and (8.13) are true for all $i \leq j$ (and hence also (8.14) holds); in case $j = 0$ only assume (8.14). Then we have to prove (8.12) for $i = j + 1$ and (8.13) in case $i = j + 1 \geq 2$: Firstly, (8.11) provides the bound for lemma 8.6 (ii) that allows to use the estimates with the constant C_2 fixed above : In case $j = 0$

$$\begin{aligned}\|\xi_0\|_{W^{2,p}} &\leq C_1\big(\|\mathrm{d}^*_{\hat{A}}(A_0 - \hat{A})\|_p + \| *(A_0 - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\ &\leq C_1 C_3 \|A - \hat{A}\|_{W^{1,p}} \\ &\leq C_1 C_3 c_0 \;=\; c_2,\end{aligned}$$

and for the case $j \geq 1$ use (8.12) and choose $\delta \leq 2C_3 c_0 C_I^{-1}$ such that

$$\begin{aligned}\|\xi_j\|_{W^{2,p}} &\leq C_1\big(\|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p + \| *(A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\ &\leq 2^{-i} C_1 C_I \|A - \hat{A}\|_q \\ &\leq \tfrac{1}{2} C_1 C_I \delta \;\leq\; c_2.\end{aligned}$$

Now since $\mathrm{d}^*_{\hat{A}}\hat{A} = \mathrm{d}^*_{\hat{A}}\big(\mathrm{d}_{\hat{A}}\xi_j + A_j\big)$ and $*\hat{A}|_{\partial M} = *(\mathrm{d}_{\hat{A}}\xi_j + A_j)|_{\partial M}$ we can rewrite

$$\begin{aligned}\mathrm{d}^*_{\hat{A}}(A_{j+1} - \hat{A}) &= \mathrm{d}^*_{\hat{A}}\big(\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big) + \mathrm{d}^*_{\hat{A}}\big[A_j - \hat{A}, \xi_j\big], \\ *\big(A_{j+1} - \hat{A}\big)|_{\partial M} &= *\big(\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big)|_{\partial M} + \big[*\big(A_j - \hat{A}\big)|_{\partial M}, \xi_j\big].\end{aligned} \tag{8.16}$$

The first terms in both right hand side expressions are estimated by lemma 8.6 (ii) and with the help of (8.6), (8.11), and (8.14) :

$$\begin{aligned}&\big\|\mathrm{d}^*_{\hat{A}}\big(\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big)\big\|_p + \big\|*\big(\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big)|_{\partial M}\big\|_{W^{1,p}_\partial} \\ &\leq C_3 \big\|\exp(\xi_j)^* A_j - A_j - \mathrm{d}_{A_j}\xi_j\big\|_{W^{1,p}} \\ &\leq C_2 C_3 \big(1 + \|A_j - \hat{A}\|_{W^{1,p}}\big)\|\xi_j\|_{W^{1,q}}\|\xi_j\|_{W^{2,p}} \\ &\leq C_1^2 C_2 C_3 (1 + 2c_0)\|A_j - \hat{A}\|_q \big(\|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p + \| *(A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big)\end{aligned}$$

Now consider the upper second terms in (8.16). Firstly, from the local formula (A.9) for $\mathrm{d}^*_{\hat{A}}$ and the Jacobi identity one obtains

$$\mathrm{d}^*_{\hat{A}}[A_j - \hat{A}, \xi_j] = [\mathrm{d}^*_{\hat{A}}(A_j - \hat{A}), \xi_j] - \big\langle A_j - \hat{A}\,,\, \mathrm{d}_{\hat{A}}\xi_j\big\rangle.$$

As in the proof of lemma 8.6 let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, then the Sobolev inequality for $W^{2,p} \hookrightarrow W^{1,r}$ holds. Thus from (8.11) and with a finite constant C arising from several Sobolev constants one obtains

$$\begin{aligned}\big\|\mathrm{d}^*_{\hat{A}}[A_j - \hat{A}, \xi_j]\big\|_p &\le \|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p \|\xi_j\|_\infty + \|A_j - \hat{A}\|_q \|\mathrm{d}_{\hat{A}}\xi_j\|_r \\ &\le C\big(\|\xi_j\|_{W^{1,q}} \|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p + \|A_j - \hat{A}\|_q \|\xi_j\|_{W^{2,p}}\big) \\ &\le CC_1 \|A_j - \hat{A}\|_q \big(\|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p + \| * (A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big)\end{aligned}$$

For the lower second term in (8.16) use (8.11) and lemma B.3 with $r = p$ and $s = q$ to obtain a constant C such that

$$\begin{aligned}\big\|[*(A_j - \hat{A})|_{\partial M}, \xi_j]\big\|_{W^{1,p}_\partial} &\le C\| * (A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \|\xi_j\|_{W^{1,q}} \\ &\le CC_1 \| * (A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \|A_j - \hat{A}\|_q.\end{aligned}$$

Now we have considered all terms in (8.16) and found a finite constant C_4 depending on c_0, C_1, C_2, C_3, and some Sobolev constants C such that

$$\begin{aligned}&\|\mathrm{d}^*_{\hat{A}}(A_{j+1} - \hat{A})\|_p + \| * (A_{j+1} - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \\ &\qquad \le C_4 \|A_j - \hat{A}\|_q \big(\|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p + \| * (A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\ &\qquad \le 2^{-j} C_I C_4 (1 + CC_I)\delta \|A - \hat{A}\|_q \\ &\qquad \le 2^{-(j+1)} C_I \|A - \hat{A}\|_q\end{aligned}$$

In the further steps we used (8.12) for $i = j$ and (8.15), and we made the possibly even smaller choice $\delta \le C_4^{-1}(1 + CC_I)^{-1}$. Since we used (8.12) this only holds for $j \ge 1$; in case $j = 0$ one has to use (8.6) and (8.1) to obtain

$$\begin{aligned}&\|\mathrm{d}^*_{\hat{A}}(A_1 - \hat{A})\|_p + \| * (A_1 - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial} \\ &\qquad \le C_4 \|A_0 - \hat{A}\|_q \big(\|\mathrm{d}^*_{\hat{A}}(A_0 - \hat{A})\|_p + \| * (A_0 - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\ &\qquad \le C_3 C_4 \|A - \hat{A}\|_q \|A - \hat{A}\|_{W^{1,p}} \\ &\qquad \le c_0 C_3 C_4 \|A - \hat{A}\|_q.\end{aligned}$$

In both cases, this proves the induction step for (8.12); where in the step for $j = 0$ the constant C_I is fixed as $C_I = c_0 C_3 C_4$.

Furthermore, (8.13) is shown in case $j + 1 \ge 2$ with the help of lemma 8.6 (ii), (8.14), (8.11), and again (8.12) for $i = j \ge 1$:

$$\begin{aligned}\|A_{j+1} - A_j\|_{W^{1,p}} &\le C_2\big(1 + \|A_j - \hat{A}\|_{W^{1,p}}\big)\|\xi_j\|_{W^{2,p}} \\ &\le C_1 C_2 (1 + 2c_0)\big(\|\mathrm{d}^*_{\hat{A}}(A_j - \hat{A})\|_p + \| * (A_j - \hat{A})|_{\partial M}\|_{W^{1,p}_\partial}\big) \\ &\le 2^{-j} C_I C_1 C_2 (1 + 2c_0) \|A - \hat{A}\|_q.\end{aligned}$$

This proves the induction step for (8.13) with $C_{II} = \frac{1}{2} C_I C_1 C_2 (1 + 2c_0)$. So we have proved (8.12) and (8.13) by induction.

Now (8.13) shows that the A_i form a $W^{1,p}$-Cauchy sequence. Indeed, for all $k > j \geq 1$

$$\|A_k - A_j\|_{W^{1,p}} \;\leq\; \sum_{i=j+1}^{k} \|A_i - A_{i-1}\|_{W^{1,p}} \;\leq\; \sum_{i=j+1}^{k} 2^{-i} C_I \|A - \hat{A}\|_q \;\leq\; 2^{-j}\delta C_I.$$

Since $\mathcal{A}^{1,p}(P)$ is a Banach space this implies that the A_i converge in the $W^{1,p}$-norm to some $A_\infty \in \mathcal{A}^{1,p}(P)$. By continuity this limit connection also satisfies (8.14) and (8.15), hence one obtains a constant $C_{CG} = 1 + CC_{II}$ (where C is the Sobolev constant for the embedding $W^{1,p} \hookrightarrow L^q$) such that

$$\|A_\infty - \hat{A}\|_{W^{1,p}} \leq C_{CG}\|A - \hat{A}\|_{W^{1,p}},$$
$$\|A_\infty - \hat{A}\|_q \leq C_{CG}\|A - \hat{A}\|_q.$$

From (8.12) one sees that

$$\mathrm{d}^*_{\hat{A}}(A_\infty - \hat{A}) = \lim_{i\to\infty} \mathrm{d}^*_{\hat{A}}(A_i - \hat{A}) \quad = 0,$$
$$*(A_\infty - \hat{A})\big|_{\partial M} = \lim_{i\to\infty} *\big(A_i - \hat{A}\big)\big|_{\partial M} = 0.$$

So it remains to show that $A_\infty = u^*A$ for some $u \in \mathcal{G}^{2,p}(P)$. For that purpose consider the sequence $u_i = \exp(\xi_1)\ldots\exp(\xi_i)$. By lemma A.5 it lies in $\mathcal{G}^{2,p}(P)$, and it satisfies $u_i^*A = A_i$. Now lemma A.8 applies since A_i converges in the $W^{1,p}$-norm and A is uniformly $W^{1,p}$-bounded anyway. Thus there exists a subsequence of the u_i that converges in the $\mathcal{C}^0$-norm to some $u \in \mathcal{G}^{2,p}(P)$. For the same subsequence (again labelled by i) $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in the L^{2p}-norm. Now $u^*A = A_\infty$ since this is the unique L^{2p}-limit of the sequence

$$u_i^{-1}Au_i + u_i^{-1}\mathrm{d}u_i \;=\; u_i^*A \;=\; A_i.$$

Thus u is the required gauge transformation that puts A in relative Coulomb gauge. □

Chapter 9

Yang-Mills Connections

This chapter introduces Yang-Mills connections in the weak and strong sense on manifolds with boundary and establishes the basic regularity results and estimates.

We consider a principal G-bundle $P \to M$, where G is a compact Lie group and M is a Riemannian n-manifold. The **Yang-Mills functional** is given by

$$\mathcal{YM}(A) = \int_M |F_A|^2$$

for smooth connections $A \in \mathcal{A}(P)$ with compact support. Here the norm $|\cdot|$ on the fibres of $\Lambda^2 \mathrm{T}^* M \otimes \mathfrak{g}_P$ is determined by the metric of M and the inner product on $\mathfrak{g}$ as in (A.7). If the base manifold M is compact then the Yang-Mills functional can be extended to $\mathcal{A}^{1,p}(P)$ for $2 \leq p < \infty$ such that $p \geq \frac{4n}{4+n}$. (The latter condition results from the Sobolev embedding $W^{1,p} \hookrightarrow L^4$ which ensures that $[A \wedge A]$ and hence F_A is of class L^2.) The Euler-Lagrange equations for the Yang-Mills functional are

$$\begin{cases} \mathrm{d}_A^* F_A = 0, \\ *F_A|_{\partial M} = 0, \end{cases} \tag{9.1}$$

Here d_A^* denotes the formally adjoint differential operator of d_A, see (A.9). Solutions of this boundary value problem are called **Yang-Mills connections**. It is not a priori clear that every critical point of the Yang-Mills functional on $\mathcal{A}^{1,p}(P)$ actually solves the (strong) Yang-Mills equation (9.1). In general, the critical points are only weak Yang-Mills connections in the following sense.

Definition 9.1 *Let $1 \leq p < \infty$ be such that $p > \frac{n}{2}$, and in case $n = 2$ assume in addition $p \geq \frac{4}{3}$. Then a connection $A \in \mathcal{A}^{1,p}_{loc}(P)$ is called a* **weak Yang-Mills connection** *if it satisfies*

$$\int_M \langle F_A , \mathrm{d}_A \beta \rangle = 0 \qquad \forall \beta \in \Omega^1(M; \mathfrak{g}_P). \tag{9.2}$$

If M is noncompact, then the test 1-forms β are required to have compact support.

Note that the Yang-Mills functional is not necessarily defined or finite for weak Yang-Mills connections. (For $n \leq 3$ we do not assume $p \geq 2$.) However, in order for the weak equation (9.2) to make sense, the assumptions on the regularity of A should at least ensure $\langle F_A , \mathrm{d}_A\beta \rangle \in L^1_{\mathrm{loc}}(M)$. The curvature F_A is locally of class L^p since the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$ holds for $p \geq \frac{n}{2}$. So $\mathrm{d}_A\beta$ should be of class $L^{p^*}_{\mathrm{loc}}$, i.e. we need the Sobolev embedding $W^{1,p} \hookrightarrow L^{p^*}$ for $\frac{1}{p^*} + \frac{1}{p} = 1$. The condition for the latter is $p \geq \frac{2n}{n+1}$. For $n = 1$ this is met due to $p \geq 1$ and for $n \geq 3$ this holds since $p \geq \frac{n}{2}$. Just for $n = 2$ this requires the additional assumption $p \geq \frac{4}{3}$. The strict inequality $p > \frac{n}{2}$ will be needed in the next lemma to show that the weak Yang-Mills equation (9.2) is preserved under the gauge action of $\mathcal{G}^{2,p}_{loc}(P)$.

Lemma 9.2 *Let $A \in \mathcal{A}^{1,p}_{loc}(P)$ be a weak Yang-Mills connection and fix a compact subset $K \subset M$. Then for every gauge transformation $u \in \mathcal{G}^{2,p}(P|_K)$ the connection $u^*A|_K \in \mathcal{A}^{1,p}(P|_K)$ also solves (9.2) for all $\beta \in \Omega^1(M;\mathfrak{g}_P)$ supported in K.*

In particular, $u^*A \in \mathcal{A}^{1,p}_{loc}(P)$ also is a weak Yang-Mills connection for every $u \in \mathcal{G}^{2,p}_{loc}(P)$.

Proof: We have seen before that F_A is of class L^p. Moreover, d_A continuously maps the space of 1-forms $W^{2,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ to $L^{p^*}(M, \Lambda^2\mathrm{T}^*M \otimes \mathfrak{g}_P)$. This can be seen from the local formula $\left(\mathrm{d}_A\beta\right)_\alpha = \mathrm{d}\beta_\alpha + [A_\alpha \wedge \beta_\alpha]$ and the Sobolev embeddings $W^{1,p} \hookrightarrow L^{p^*}$ and $W^{2,p} \hookrightarrow \mathcal{C}^0$. (For the latter we need the strict inequality $p > \frac{n}{2}$). Due to these regularities (9.2) in fact holds for all $\tilde{\beta} \in W^{2,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$.

In particular, let $K \subset M$ be a compact set and let $u \in \mathcal{G}^{2,p}(P|_K)$. Then for every smooth test 1-form β supported in K one can extend $\tilde{\beta} := u\beta u^{-1}$ by 0 outside of K to obtain such a 1-form of class $W^{2,p}$ – see the lemmata A.5 and A.7. Now (9.2) holds for $\tilde{\beta}$ and thus

$$\int_M \langle F_{u^*A} , \mathrm{d}_{u^*A}\beta \rangle \;=\; \int_M \left\langle u^{-1}F_A\, u \,,\, u^{-1}\mathrm{d}_A\tilde{\beta}\, u\right\rangle \;=\; \int_M \left\langle F_A \,,\, \mathrm{d}_A\tilde{\beta}\right\rangle \;=\; 0.$$

Here we used (A.4) and the fact that in all local trivializations

$$\begin{aligned}\left(\mathrm{d}_A(u\beta u^{-1})\right)_\alpha &= u_\alpha \mathrm{d}\beta_\alpha u_\alpha^{-1} + u_\alpha[u_\alpha^{-1}A_\alpha u_\alpha \wedge \beta_\alpha]u_\alpha^{-1} + u_\alpha[u_\alpha^{-1}\mathrm{d}u_\alpha \wedge \beta_\alpha]u_\alpha^{-1} \\ &= \left(u\,\mathrm{d}_{u^*A}\beta\, u^{-1}\right)_\alpha.\end{aligned}$$

□

The next lemma shows that (9.1) and (9.2) are equivalent for connections of sufficiently high regularity, and hence the strong Yang-Mills equation also is preserved under (sufficiently regular) gauge transformations.

Lemma 9.3 *Let $1 \leq p < \infty$ be such that $p \geq \frac{2n}{n+2}$. Fix $A \in \mathcal{A}^{1,p}_{loc}(P)$ and two equivariant forms $\omega \in W^{1,p}_{loc}(M, \Lambda^k\mathrm{T}^*M \otimes \mathfrak{g}_P)$ and $\gamma \in L^p_{loc}(M, \Lambda^{k-1}\mathrm{T}^*M \otimes \mathfrak{g}_P)$ for some $k \in \mathbb{N}$. Then the following is equivalent:*

(i) For all smooth $\eta \in \Omega^{k-1}(M;\mathfrak{g}_P)$ with compact support

$$\int_M \langle\, \omega\,,\, \mathrm{d}_A\eta\,\rangle = \int_M \langle\, \gamma\,,\, \eta\,\rangle\,.$$

(ii)

$$\begin{cases} \mathrm{d}_A^*\omega = \gamma, \\ *\omega|_{\partial M} = 0. \end{cases}$$

Proof: The Sobolev embedding $W^{1,p} \hookrightarrow L^2$ (which holds due to $p \geq \frac{2n}{n+2}$) ensures that $\langle\, [A\wedge\eta]\wedge *\omega\,\rangle$ can be integrated over P for all smooth $\eta \in \Omega^{k-1}(P;\mathfrak{g})$ with compact support. Thus Stokes' theorem and (A.5) yield

$$\begin{aligned} &\int_M \langle\, \omega\,,\, \mathrm{d}_A\eta\,\rangle \\ &= \int_{M\cap \mathrm{supp}\,\eta} \langle\, \mathrm{d}\eta\wedge *\omega\,\rangle + \int_M \langle\, [A\wedge\eta]\wedge *\omega\,\rangle \\ &= \int_{M\cap \mathrm{supp}\,\eta} \langle\, \eta\wedge *\mathrm{d}^*\omega\,\rangle + \int_{\partial(M\cap \mathrm{supp}\,\eta)} \langle\, \eta\wedge *\omega\,\rangle - (-1)^{k-1}\int_M \langle\, \eta\wedge [A\wedge *\omega]\,\rangle \\ &= \int_M \langle\, \mathrm{d}_A^*\omega\,,\, \eta\,\rangle + \int_{\partial M} \langle\, \eta\wedge *\omega\,\rangle\,. \end{aligned}$$

First assume (i). Then this identity holds without the boundary term for all η that vanish on ∂M and proves that $\mathrm{d}_A^*\omega = \gamma$. The above identity thus becomes $\int_{\partial M} \langle\, \eta\wedge *\omega\,\rangle = 0$, which implies that $*\omega|_{\partial M} = 0$. If on the other hand (ii) holds then the above identity proves (i) directly. □

The main goal of this chapter is to establish the following regularity result for Yang-Mills connections on compact base manifolds and such noncompact base manifolds that are exhausted by compact deformation retracts. Again we only consider exhausting sequences $M = \bigcup_{k\in\mathbb{N}} M_k$ such that each M_k lies in the interior of M_{k+1}.

Theorem 9.4 *Let $1 < p < \infty$ be such that $p > \frac{n}{2}$, and in case $n = 2$ assume in addition $p \geq \frac{4}{3}$. Then the following holds.*

(i) Assume that M is compact. Then for every weak Yang-Mills connection $A \in \mathcal{A}^{1,p}(P)$ there exists a gauge transformation $u \in \mathcal{G}^{2,p}(P)$ such that u^*A is smooth.

(ii) Assume that $M = \bigcup_{k\in\mathbb{N}} M_k$ is exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Then for every weak Yang-Mills connection $A \in \mathcal{A}^{1,p}_{loc}(P)$ there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{loc}(P)$ such that u^*A is smooth.

To prove this theorem one uses the local slice theorem 8.1 and then needs some regularity result for Yang-Mills connections in relative Coulomb gauge. For $k \geq 2$ this will be the following result: If $\tilde{A} \in \mathcal{A}(P)$ is a smooth reference connection and $A = \tilde{A} + \alpha \in \mathcal{A}^{k,p}(P)$ such that

$$\begin{cases} \mathrm{d}_{\tilde{A}}^* \alpha &= 0, \\ *\alpha|_{\partial M} &= 0, \end{cases} \qquad\qquad \begin{cases} \mathrm{d}_A^* F_A &= 0, \\ *F_{\tilde{A}}|_{\partial M} &= 0 \end{cases} \tag{9.3}$$

then in fact $A \in \mathcal{A}^{k+1,p}(P)$. The main reason for this regularity is that

$$\begin{aligned} \mathrm{d}_{\tilde{A}}^* \mathrm{d}_{\tilde{A}} \alpha &= -\mathrm{d}_{\tilde{A}}^* F_{\tilde{A}} - \tfrac{1}{2} \mathrm{d}_{\tilde{A}}^* [\alpha \wedge \alpha] + (-1)^n * [\alpha \wedge *F_A] \\ &\in W^{k-1,p}(\mathrm{T}^* M \otimes \mathfrak{g}_P). \end{aligned}$$

Hence also $\Delta_{\tilde{A}} \alpha$ is of class $W^{k-1,p}$, where $\Delta_{\tilde{A}} = \mathrm{d}_{\tilde{A}}^* \mathrm{d}_{\tilde{A}} + \mathrm{d}_{\tilde{A}} \mathrm{d}_{\tilde{A}}^*$ is the generalized Hodge Laplace operator for 1-forms on the vector bundle $\mathfrak{g}_P$ with the connection $\tilde{A}$. So the regularity of A is a consequence of a generalization of the regularity theorem 5.2 for scalar 1-forms.

However, if only $A \in \mathcal{A}^{1,p}(P)$ then the Yang-Mills equation has to be used in its weak form. Due to the nonlinearities one also has to be careful with the Sobolev exponent. Moreover, one has to use cutoff functions to obtain regularity results in the case of a noncompact manifold. This leads to inhomogenous equations and even inhomogenous boundary conditions instead of (9.3). So there is some work to be done in order to deduce the regularity of Yang-Mills connections (in relative Coulomb gauge) from the regularity result for scalar 1-forms.

Furthermore, together with these regularity results we provide the corresponding estimates that will be used for the proof of the strong Uhlenbeck compactness theorem E and E′ in chapter 10. Since theorem E′ concerns sequences of Yang-Mills connections with respect to sequences of metrics, we also have to take the metric on M into account when establishing the estimates.

We will prove theorem 9.4 by iteration of the regularity results in corollary 9.6. These are easy consequences of the basic componentwise regularity result for trivial bundles in proposition 9.5. The latter is stated in high generality in order to allow the application for a larger class of boundary conditions for the Yang-Mills equations. For example, in [W] one uses this result to obtain regularity for certain components of the connection when there is a restriction on the test 1-forms β in the weak Yang-Mills equation (9.2). Moreover, the componentwise formulation of this basic regularity result evades explicit calculations with cutoff functions – they can be included in the vector field.

In the following proposition 9.5, (U, g) is a compact Riemannian n-manifold with (possibly empty) boundary ∂U, and ν denotes the outer unit normal to ∂U. This proposition relies on the componentwise regularity result for scalar 1-forms in theorem 5.5. So we introduce the following notation in analogy to chapter 5:

$$\begin{aligned} \mathcal{C}_\delta^\infty(U, \mathfrak{g}) &:= \bigl\{ \phi \in \mathcal{C}^\infty(U, \mathfrak{g}) \bigm| \phi|_{\partial U} = 0 \bigr\}, \\ \mathcal{C}_\nu^\infty(U, \mathfrak{g}) &:= \bigl\{ \phi \in \mathcal{C}^\infty(U, \mathfrak{g}) \bigm| \tfrac{\partial \phi}{\partial \nu}|_{\partial U} = 0 \bigr\}. \end{aligned}$$

Proposition 9.5 *Fix a smooth reference connection $\tilde{A} \in \mathcal{A}(U \times G)$ of the trivial G-bundle over U. Let $X \in \Gamma(\mathrm{T}U)$ be a smooth vector field that is either perpendicular to the boundary, i.e. $X|_{\partial U} = \psi \cdot \nu$ for some $\psi \in \mathcal{C}^\infty(\partial U)$, or is tangential, i.e. $X|_{\partial U} \in \Gamma(\mathrm{T}\partial U)$. In the first case let $\mathcal{T} = \mathcal{C}^\infty_\delta(U, \mathfrak{g})$, in the latter case let $\mathcal{T} = \mathcal{C}^\infty_\nu(U, \mathfrak{g})$. Moreover, let $N \subset \partial U$ be an open subset such that X vanishes in a neighbourhood of $\partial U \setminus N \subset U$.*

Let $1 < p < \infty$ and $k \in \mathbb{N}$ be such that either $kp > n$, or $k = 1$ and $\frac{n}{2} < p < n$ (if $n = 2$ assume in addition $p \geq \frac{4}{3}$). In the first case let $q := p$, in the latter case let $q := \frac{np}{2n-p}$. Then there exists a constant C such that the following holds.

Let $A = \tilde{A} + \alpha \in \mathcal{A}^{k,p}(U \times G)$ be a connection. Suppose that it satisfies

$$\begin{cases} \mathrm{d}^*_{\tilde{A}}\alpha = 0, \\ *\alpha|_{\partial U} = 0 \quad \text{on } N, \end{cases}$$

and that for all $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T}$

$$\int_U \langle F_A \, , \, \mathrm{d}_A \beta \rangle = 0.$$

Then $\alpha(X) \in W^{k+1,q}(U, \mathfrak{g})$ and

$$\|\alpha(X)\|_{W^{k+1,q}} \leq C \left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|^3_{W^{k,p}}\right).$$

Moreover, the constant C can be chosen such that it depends continuously on the metric and the vector field X with respect to the $W^{k+1,\infty}$-topology.

Corollary 9.6 *Let $1 < p < \infty$ and $k \in \mathbb{N}$ be such that either $kp > n$, or $k = 1$ and $\frac{n}{2} < p < n$ (if $n = 2$ assume in addition $p \geq \frac{4}{3}$). In the first case let $q := p$, in the latter case let $q := \frac{np}{2n-p}$.*

(i) Assume M is compact and let $\tilde{A} \in \mathcal{A}(P)$ be a smooth reference connection. Then there exists a constant C such that the following holds.

Suppose that the connection $A = \tilde{A} + \alpha \in \mathcal{A}^{k,p}(P)$ satisfies

$$\begin{cases} \mathrm{d}^*_{\tilde{A}}\alpha = 0, \\ *\alpha|_{\partial M} = 0, \end{cases}$$

and for all smooth $\beta \in \Omega^1(M; \mathfrak{g}_P)$

$$\int_M \langle F_A \, , \, \mathrm{d}_A \beta \rangle = 0.$$

Then $A \in \mathcal{A}^{k+1,q}(P)$ and

$$\|\alpha\|_{W^{k+1,q}} \leq C \left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|^3_{W^{k,p}}\right).$$

(ii) *Let $\tilde{A} \in \mathcal{A}(P)$ be a smooth reference connection over the possibly noncompact manifold M. Let $M'' \subset M' \subset M$ be compact submanifolds such that M'' is contained in the interior of M'. Then there exists a constant C such that the following holds: Let $A = \tilde{A} + \alpha \in \mathcal{A}^{k,p}(P|_{M'})$ be a connection over M'. Suppose that it satisfies*

$$\begin{cases} \mathrm{d}^*_{\tilde{A}}\alpha = 0 & \text{on } M', \\ *\alpha|_{\partial M'} = 0 & \text{on } \partial M \cap \partial M', \end{cases}$$

and that for all smooth $\beta \in \Omega^1(M; \mathfrak{g}_P)$ supported in M'

$$\int_M \langle F_A \,,\, \mathrm{d}_A\beta \rangle = 0.$$

Then $A|_{M''} \in \mathcal{A}^{k+1,q}(P|_{M''})$ and

$$\|\alpha\|_{W^{k+1,q}(M'')} \le C\Big(1 + \|\alpha\|_{W^{k,p}(M')} + \|\alpha\|^3_{W^{k,p}(M')}\Big).$$

(iii) *The constants C in (i) and (ii) can be chosen such that they depend continuously on the metric with respect to the $W^{k+1,\infty}$-topology.*

Note that the assumptions on p in both the above proposition and corollary ensure $p \ge \frac{2n}{n+1}$ (as in definition 9.1), and thus $q \ge 1$. This type of regularity result will be iterated in the proof of theorem 9.4 (i) and (ii) as well as theorem E and E$'$. The general proceeding is the same in all cases:

For $k \ge 2$ or $p > n$ one directly gets from $W^{k,p}$-regularity or -bounds to $W^{k+1,p}$-regularity or -bounds. If $p < n$ then one needs a separate iteration to get from $W^{1,p}$ to $W^{2,p}$. This is since from $W^{1,p}$ one only gets to $W^{2,q}$ with $q = \frac{np}{2n-p} < p$. The Sobolev embedding $W^{2,q} \hookrightarrow W^{1,p'}$ then gives $W^{1,p'}$-regularity with $p' = \frac{nq}{n-q}$. By iterating this argument one obtains W^{1,p_i}- and W^{2,q_i}-regularity for the sequences defined in the subsequent lemma.

If the latter iteration reaches $W^{1,n}$ (or actually starts there), then we can get to $W^{2,n}$ (and thus to $W^{1,p}$ if we started with $p < n$) as follows: We just assume $W^{1,\frac{3}{4}n}$-regularity, from which the iteration as above gives regularity in $W^{2,\frac{3}{5}n}$, $W^{1,\frac{3}{2}n}$, and thus finally in $W^{2,\frac{3}{2}n}$, which embeds into $W^{2,n}$.

The following lemma shows that the iteration described above (starting from $k = 1$ and $p \le n$) indeed reaches $W^{2,p}$ after finitely many steps.

Lemma 9.7 *Assume $\frac{n}{2} < p \le n$. Define sequences (p_i) and (q_i) by $p_0 := p$ and for all $i \in \mathbb{N}_0$*

$$q_i := \begin{cases} \frac{np_i}{2n-p_i} & ;\text{ if } p_i < n, \\ p_i & ;\text{ if } p_i \ge n. \end{cases}$$

In case $p_i \ge n$ terminate the sequence with this $q_i = p_i$; in case $p_i < n$ let

$$p_{i+1} := \frac{nq_i}{n - q_i}.$$

This defines a finite increasing sequence (p_i) that terminates with some $q_j \ge p$.

Proof: Firstly, this sequence obviously does not terminate unless $p_i \geq n$. Secondly, q_i is always welldefined. Now if the sequences do not terminate at q_i, then $p_i < n$, thus $q_i < n$, and so p_{i+1} is also welldefined. Moreover, one sees inductively that

$$p_{i+1} = \frac{nq_i}{n-q_i} = \frac{np_i}{2n-2p_i} \geq \theta p_i,$$

where $\theta := \frac{n}{2n-2p} > 1$ due to $p > \frac{n}{2}$. As start of the induction one only needs $p_0 \geq p$. Next, if $p_i \geq p$ for some $i \in \mathbb{N}_0$, then $\frac{n}{2n-2p_i} \geq \theta$ and thus above calculation gives $p_{i+1} \geq \theta p_i > p_i \geq p$.

So the sequence p_i grows at a rate of at least $\theta > 1$ until it reaches $p_j \geq n$ for some finite $j \in \mathbb{N}$ and terminates after $q_j \geq n \geq p$. □

Proof of proposition 9.5 :
Note that in any case we have $p \geq q$; the corresponding inequality between the L^p- and L^q-norm and the embedding $W^{k,p} \hookrightarrow W^{k,q}$ will be used without further mention in the following. Also note that all derivatives of $\tilde{A}$ are bounded since U is compact.

Let $A = \tilde{A} + \alpha \in W^{k,p}(U, \mathrm{T}^*U \otimes \mathfrak{g})$ be a solution of the given equations. Then we will prove that α satisfies the assumptions of theorem 5.5 on U. This theorem is stated for scalar 1-forms. However, it generalizes straightforward to $\mathfrak{g}$-valued differential forms. To see this decompose the differential forms with respect to an orthonormal basis of $\mathfrak{g}$. Then the theorem applies to each of these components separately and implies the overall regularity.

Firstly, we have the boundary condition $*\alpha|_{\partial U} = 0$ on N. Secondly,

$$\mathrm{d}^*\alpha = *[\tilde{A} \wedge *\alpha] =: G \in W^{k,q}(U, \mathfrak{g}) \qquad \text{with} \qquad \|G\|_{W^{k,q}} \leq C\|\alpha\|_{W^{k,p}}.$$

Here and in the following C denotes any finite constant. These possibly depend on $\tilde{A}$ and we will make sure that they depend continuously on the metric with respect to the $W^{k+1,\infty}$-topology. For the weak equation on α consider any $\beta = \phi \cdot \iota_X g$ with $\phi \in \mathcal{T}$, then

$$\begin{aligned}
&\int_U \langle \mathrm{d}\alpha\,,\, \mathrm{d}\beta \rangle \\
&= \int_U \langle F_A\,,\, \mathrm{d}_A\beta \rangle - \int_U \langle F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\,,\, \mathrm{d}_A\beta \rangle - \int_U \langle \mathrm{d}\alpha\,,\, [A \wedge \beta] \rangle \\
&= -\int_U \langle \mathrm{d}_A^*\big(F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\big)\,,\, \beta \rangle - \int_{\partial U} \langle \beta \wedge *\big(F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\big) \rangle \\
&\quad + (-1)^n \int_U \langle *[A \wedge *\mathrm{d}\alpha]\,,\, \beta \rangle \\
&= \int_U \langle \gamma\,,\, \beta \rangle + \int_{\partial U} \langle \beta \wedge *\omega \rangle.
\end{aligned}$$

This uses partial integration (and smooth approximation) as in lemma 4.1 for $\nabla = \mathrm{d}_{\tilde{A}+\alpha}$ on the bundle $E = U \times \mathfrak{g}$. In the final expression we have introduced

$\gamma \in W^{k-1,q}(U, \mathrm{T}^*U \otimes \mathfrak{g})$ and $\omega \in W^{k,q}(U, \Lambda^2\mathrm{T}^*U \otimes \mathfrak{g})$ which satisfy the following estimates:

$$\begin{aligned}
&\|\gamma\|_{W^{k-1,q}} \\
&= \big\| -\mathrm{d}_A^*\big(F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\big) + (-1)^n * [A \wedge *\mathrm{d}\alpha] \big\|_{W^{k-1,q}} \\
&\le \big\| \mathrm{d}_{\tilde{A}}^*\big(F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\big) \big\|_{W^{k-1,q}} + \big\| [\alpha \wedge *\big(F_{\tilde{A}} + [(\tilde{A} + \alpha) \wedge \alpha]\big) \big\|_{W^{k-1,q}} \\
&\qquad + \big\| [(\tilde{A} + \alpha) \wedge *\mathrm{d}\alpha] \big\|_{W^{k-1,q}} \\
&\le C\big(\, \big\|F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\big\|_{W^{k,q}} + \|\alpha\|_{W^{k-1,r}} \big\|F_{\tilde{A}} + [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha]\big\|_{W^{k-1,p}} \\
&\qquad + \big\|\tilde{A} + \alpha\|_{W^{k-1,r}} \|\mathrm{d}\alpha\|_{W^{k-1,p}} \, \big) \\
&\le C \left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}^2 + \|\alpha\|_{W^{k,p}}^3 \right),
\end{aligned}$$

$$\begin{aligned}
\|\omega\|_{W^{k,q}} \;&=\; \big\| -F_{\tilde{A}} - [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha] \big\|_{W^{k,q}} \;\le\; \|F_{\tilde{A}}\|_{W^{k,q}} + \big\| [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha] \big\|_{W^{k,q}} \\
&\le\; C \left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}^2 \right).
\end{aligned}$$

Here we have used lemma B.3 with p replaced by q and $r = s = p$ (note that in case $k = 1$ and $p < n$ we have $\frac{1}{p} + \frac{1}{p} = \frac{1}{q} + \frac{1}{n}$ and $p > q$) to obtain

$$\big\| [(\tilde{A} + \tfrac{1}{2}\alpha) \wedge \alpha] \big\|_{W^{k,q}} \;\le\; C\|\tilde{A} + \tfrac{1}{2}\alpha\|_{W^{k,p}} \|\alpha\|_{W^{k,p}} \;\le\; C\big(\|\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}^2 \big).$$

The same bound holds for $\big\| [(\tilde{A} + \frac{1}{2}\alpha) \wedge \alpha] \big\|_{W^{k-1,p}}$ by the Sobolev estimate for $W^{k,q} \hookrightarrow W^{k-1,p}$ (in case $q \neq p$ this is due to $\frac{1}{q} = \frac{2}{p} - \frac{1}{n} \le \frac{1}{p} - \frac{1}{n}$).

Moreover, in case $k \ge 2$ we used lemma B.3 with (k, p) replaced by $(k-1, q)$ and for $s = p$ and $1 < r < \infty$ such that there is a Sobolev embedding $W^{k,p} \hookrightarrow W^{k-1,r}$. Such r has to satisfy

$$\tfrac{k-1}{n} \;>\; \tfrac{1}{r} \;\ge\; \tfrac{1}{p} - \tfrac{1}{n},$$

and it exists since $\frac{1}{p} < \frac{k}{n}$. In case $k = 1$ we used the Hölder inequality for $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$, that is $r = \infty$ in case $p > n$ and $r = \frac{np}{n-p}$ in case $p < n$. Again, the Sobolev embedding $W^{1,p} \hookrightarrow L^r$ holds in both cases.

All constants in above estimates are independent of the vector field X and depend on the metric only in as much as the involved Sobolev norms depend on the metric. Since we only considered Sobolev norms of order up to k, they all depend continuously on the metric even with respect to the $W^{k,\infty}$-topology.

So we have established that α satisfies the assumptions of theorem 5.5 on the manifold U with the inhomogenities γ, ω, and G. This theorem now asserts that $\alpha(X) \in W^{k+1,q}(U, \mathfrak{g})$ and for some constant C_X

$$\begin{aligned}
\|\alpha(X)\|_{W^{k+1,q}} &\le C_X \big(\|\gamma\|_{W^{k-1,q}} + \|\omega\|_{W^{k,q}} + \|G\|_{W^{k,q}} + \|\alpha\|_{W^{k,q}} \big) \\
&\le C \left(1 + \|\alpha\|_{W^{k,p}} + \|\alpha\|_{W^{k,p}}^3 \right).
\end{aligned}$$

Here we have used the fact that $a \le \frac{1}{2}(1 + a^2)$ for all $a \in \mathbb{R}$ to remove the term $\|\alpha\|_{W^{k,p}}^2$. The final constant C results from C_X and the previous estimates. We know that C_X depends continuously on the metric and the vector field X with respect to the $W^{k+1,\infty}$-topology, and hence C also meets this continuity. $\square$

Proof of corollary 9.6 :
The idea for the proof of both (i) and (ii) is to use proposition 9.5 in finitely many bundle charts covering M or M'' respectively. By remark B.1 it suffices to prove the regularity and estimate for the local representatives of the connections in these bundle charts. The norm that arises from the Sobolev norms in these local trivializations is equivalent to the Sobolev norm on the whole bundle. Moreover, one can check that the constant in the equivalence of these Sobolev norms up to order $k+1$ depends continuously on the metric with respect to the $W^{k+1,\infty}$-topology. Hence it also suffices to prove the continuity of the constant for the estimates in the bundle charts.

For (i) choose a finite bundle atlas over an open covering by coordinate charts, $M = \bigcup_{i=1}^{N} U_i$. Here the coordinates near the boundary ∂M are chosen such that the coordinate vector fields are either tangential or normal to the boundary. One can find open subsets $V_i \subset U_i$ that still cover M but such that the closure of V_i is contained in U_i. Thus there exist cutoff functions $\psi_i \in \mathcal{C}^\infty(M)$ that are supported in U_i and satisfy $\psi_i|_{V_i} \equiv 1$. These choices are all made independently of the metric on M except for the normal coordinate vector fields near the boundary. (But for $W^{k+1,\infty}$-close metrics these vector fields can also be chosen $W^{k+1,\infty}$-close as in theorem 5.2.)

Now let $A = \tilde{A} + \alpha \in \mathcal{A}^{k,p}(P)$ be as supposed. We drop the subscript i and denote the representative of α over U again by $\alpha \in W^{k,p}(U, \mathrm{T}^*U \otimes \mathfrak{g})$. Then we have $*\alpha|_{\partial U} = 0$ on the subset $N = \partial U \cap \partial M$ of the boundary. Hence α satisfies the assumptions of proposition 9.5 with the above $N \subset \partial U$. Moreover, note that the cutoff function ψ vanishes in a neighbourhood of $\partial U \setminus N = \partial U \setminus \partial M$ since ψ is supported in U. Hence for all coordinate vector fields Y on U we can choose the vector field $X = \psi Y$ in proposition 9.5. That provides $\alpha(X) \in W^{k+1,q}(U, \mathfrak{g})$ with the according estimate on $\alpha(Y)|_V = \alpha(X)|_V \in W^{k+1,q}(V, \mathfrak{g})$. Since Y runs through all coordinate vector fields this proves the regularity and estimate for α in all bundle charts over the V_i. The global result then follows from remark B.1.

It remains to show that the constant in this local estimate meets the continuity claimed in (iii). Recall that the constant in proposition 9.5 depends continuously on the metric and the vector field with respect to the $W^{k+1,\infty}$-topology. Thus it suffices to choose $W^{k+1,\infty}$-close coordinate vector fields Y for $W^{k+1,\infty}$-close metrics to obtain the required continuity.

To prove (ii) one chooses a finite bundle atlas over coordinate charts $U_i \subset M$ as before and such that $\mathrm{int}(M') = \bigcup_{i=1}^{N} U_i$. One finds open subsets V_i whose closure is contained in U_i and that cover M'' (which is a compact subset of $\mathrm{int}(M')$). Then choose cutoff functions ψ_i as before. Now every local representative of a given solution $A = \tilde{A} + \alpha \in \mathcal{A}^{k,p}(P|_{M'})$ satisfies $*\alpha|_{\partial U} = 0$ on the subset $N = \partial U \cap \partial M$ of the boundary. (Note that ∂U and $\partial M'$ coincide near the intersections of the chart $U \subset M'$ with the boundary ∂M.) Moreover, the cutoff function ψ vanishes in a neighbourhood of $\partial U \setminus N = \partial U \setminus \partial M$. Hence for all coordinate vector fields Y on U the vector field $X = \psi Y$ is permissible for proposition 9.5. The weak equation in the proposition holds since for every $\phi \in \mathcal{C}^\infty(U, \mathfrak{g})$ the 1-form $\phi \cdot \iota_X g \in \Omega^1(U; \mathfrak{g})$ vanishes on $\partial U \setminus \partial M$. Thus it represents a 1-form $\beta \in \Omega^1(M; \mathfrak{g}_P)$ that is supported in $U \subset M'$ and so can be used in the weak Yang-Mills equation for the given connection.

So as before proposition 9.5 yields the regularity and estimate in every bundle chart over the V_i, and this adds up to the global result. Just note that one uses estimates on the U_i that are not necessarily contained in M'' and hence in the global estimate one obtains the Sobolev norms over M' on the right hand side. The continuity of the constant in (iii) is seen as for (i). □

For theorem 9.4 in the noncompact case (ii) we moreover need the following construction that uses an easier version (the subsequent lemma) of the extension argument in lemma 7.8.

Proposition 9.8 *Assume $M = \bigcup_{k \in \mathbb{N}} M_k$ is exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Let $A \in \mathcal{A}^{1,p}_{loc}(P)$ and suppose that for each $k \in \mathbb{N}$ there is a gauge transformation $u_k \in \mathcal{G}^{2,p}(P|_{M_k})$ such that $u_k^* A|_{M_k}$ is smooth. Then there exists a gauge transformation $u \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^* A$ is smooth.*

Lemma 9.9 *Let $\Omega \subset M'' \subset M'$ be compact submanifolds of M with $\Omega \subset \mathrm{int}(M'')$ and such that M'' is a deformation retract of M. Let $A \in \mathcal{A}^{1,p}(P|_{M'})$ and suppose that $u \in \mathcal{G}^{2,p}(P|_{M''})$ and $v \in \mathcal{G}^{2,p}(P|_{M'})$ are such that both $u^* A|_{M''}$ and $v^* A|_{M'}$ are smooth. Then there exists a gauge transformation $w \in \mathcal{G}^{2,p}(P|_{M'})$ such that $w|_\Omega = u$ and $w^* A \in \mathcal{A}(P|_{M'})$ is smooth.*

Proof: By lemma A.5 one has $h := v^{-1} u|_{M''} \in \mathcal{G}^{2,p}(P|_{M''})$, and this gauge transforms $v^* A$ into $u^* A$. Consider any bundle chart over some $U \subset M''$ and let h be represented by $h \in W^{2,p}(U, G)$, then as in lemma A.7

$$\nabla(h^{-1}\mathrm{d}h) = \nabla(u^* A) - h^{-1}\big(\nabla(v^* A)\big)h + \big[h^{-1}\mathrm{d}h \wedge h^{-1}(v^* A)h\big].$$

Here $u^* A$ and $v^* A$ denote the local representatives of the connections, that is smooth $\mathfrak{g}$-valued 1-forms on U. From this we can deduce that h is in fact smooth: Suppose that $h \in \mathcal{C}^\ell(U, G) \cap \mathcal{G}^{\ell+1,2p}(U)$ (which is true for $\ell = 0$ as a start due to the Sobolev embedding $W^{2,p} \hookrightarrow W^{1,2p}$). Then above equality implies that $h^{-1}\mathrm{d}h \in W^{\ell+1,2p}(U, \mathfrak{g})$, and the Sobolev embedding $W^{\ell+1,2p} \hookrightarrow \mathcal{C}^\ell(U, \mathfrak{g})$ asserts $\mathrm{d}h \in \mathcal{C}^\ell(U, \mathrm{T}^*U \otimes \mathfrak{g})$. Thus $h \in \mathcal{C}^{\ell+1}(U, G) \cap \mathcal{G}^{\ell+2,2p}$, and this can be iterated to deduce that all representatives h are smooth.

Now by lemma 7.7 (i) the smooth gauge transformation h on $P|_{M''}$ can be modified outside of $P|_\Omega$ and extended to a smooth gauge transformation $\tilde{h}$ on $P|_{M'}$. Use this to define $w := v\tilde{h} \in \mathcal{G}^{2,p}(P|_{M'})$, which satisfies $w|_\Omega = vh = u$, and also $w^* A = \tilde{h}^*(v^* A)$ is smooth on M'. □

Proof of proposition 9.8 :
We use lemma 9.9 to inductively construct the required gauge transformation on the exhausting submanifolds M_k: We set $u|_{M_2} := u_2 \in \mathcal{G}^{2,p}(P|_{M_2})$ as start of the induction, which guarantees the smoothness of $u^*A|_{M_2}$.

Now assume that $u|_{M_k} \in \mathcal{G}^{2,p}(P|_{M_k})$ is defined for some $k \geq 2$ such that $u^*A|_{M_k}$ is smooth. Then use lemma 9.9 on $\Omega \subset M'' \subset M'$ with u replaced by $u|_{M_k}$ and $v := u_{k+1}$. It provides $w \in \mathcal{G}^{2,p}(P|_{M_{k+1}})$ with $w|_{M_{k-1}} = u$ such that $w^*A|_{M_{k+1}}$ is smooth. So if we set $u|_{M_{k+1}} := w$, then this leaves u unchanged on M_{k-1} and makes $u^*A|_{M_{k+1}}$ smooth. This defines $u \in \mathcal{G}^{2,p}_{loc}(P)$ since $u|_{M_k}$ is of class $W^{2,p}$ for every $k \in \mathbb{N}$, and the construction ensures the smoothness of u^*A on M. □

Proof of theorem 9.4 :
Since $p > \frac{n}{2}$ we can fix a $p \leq q < \infty$ that meets the condition $\frac{1}{n} > \frac{1}{q} > \frac{1}{p} - \frac{1}{n}$ of theorem 8.1. Then we first prove (i):

Let a weak Yang-Mills connection $A \in \mathcal{A}^{1,p}(P)$ be given, fix a constant $c_0 > 0$, and let $\delta > 0$ be the constant from theorem 8.1 with the reference connection A. Then find a smooth connection $\tilde{A} \in \mathcal{A}(P)$ such that

$$\|\tilde{A} - A\|_q \leq \delta \qquad \text{and} \qquad \|\tilde{A} - A\|_{W^{1,p}} \leq c_0.$$

This is possible since $\|\tilde{A} - A\|_q \leq C\|\tilde{A} - A\|_{W^{1,p}}$ for some finite Sobolev constant C and since $\mathcal{A}^{1,p}(P)$ is the $W^{1,p}$-completion of the set of smooth connections. Now theorem 8.1 provides a gauge transformation $\tilde{u} \in \mathcal{G}^{2,p}(P)$ that puts $\tilde{A}$ into relative Coulomb gauge with respect to A. Then we have $u := \tilde{u}^{-1} \in \mathcal{G}^{2,p}(P)$ by lemma A.5, and lemma 8.4 asserts that $\alpha := u^*A - \tilde{A}$ meets

$$\begin{cases} \mathrm{d}^*_{\tilde{A}}\alpha = 0, \\ *\alpha|_{\partial M} = 0. \end{cases}$$

This is the first differential equation for $u^*A = \tilde{A} + \alpha$ in corollary 9.6 (i). The second (weak) equation is provided by the fact that A and hence also u^*A is a weak Yang-Mills equation (see lemma 9.2):

$$\int_M \langle\, F_{u^*A} \,,\, \mathrm{d}_{u^*A}\beta \,\rangle = 0 \qquad \forall \beta \in \Omega^1(M; \mathfrak{g}_P).$$

Now iterate corollary 9.6 (i) to prove that u^*A is smooth: As a start one has $u^*A \in \mathcal{A}^{1,p}(P)$. In case $p > n$ the corollary directly implies $u^*A \in \mathcal{A}^{2,p}(P)$ as a first step. In case $p = n$ one can replace p by some $\frac{n}{2} < p < n$ due to the compactness of M and start the iteration from that regularity. In case $p < n$ the iteration of the corollary and the Sobolev embeddings $W^{2,q_i} \hookrightarrow W^{1,p_i}$ yield $u^*A \in \mathcal{A}^{2,q_i}(P)$ for the sequences q_i, p_i defined as in lemma 9.7. Since $q_j \geq p$ for some $j \in \mathbb{N}$, one also obtains $u^*A \in \mathcal{A}^{2,p}(P)$ after finitely many iterations.

So in all cases we have proven $u^*A \in \mathcal{A}^{2,p}(P)$, where $p > \frac{n}{2}$. Now iterate corollary 9.6 (i) again to deduce $u^*A \in \mathcal{A}^{k,p}(P)$ for all $k \in \mathbb{N}$. This implies that u^*A is smooth and proves (i).

To prove (ii) one proceeds as in (i) for every compact submanifold M_{k+1} to find a smooth reference connection $\tilde{A}_k \in \mathcal{A}(P|_{M_{k+1}})$ and a gauge transformation $u_k \in \mathcal{G}^{2,p}(P|_{M_{k+1}})$ such that $\alpha_k := u_k^*A|_{M_{k+1}} - \tilde{A}_k$ satisfies the relative Coulomb gauge conditions

$$\begin{cases} \mathrm{d}^*_{\tilde{A}_k}\alpha_k = 0, \\ *\alpha_k\big|_{\partial M_{k+1}} = 0. \end{cases}$$

Moerover, A is a weak Yang-Mills connection, so lemma 9.2 asserts that u_k^*A satisfies for all test 1-forms $\beta \in \Omega^1(M;\mathfrak{g}_P)$ supported in M_{k+1}

$$\int_M \langle F_{u_k^*A} \, , \, \mathrm{d}_{u_k^*A}\beta \rangle = 0.$$

Now fix compact submanifolds $M_k \subset M_k^\ell \subset M_{k+1}$ such that $M_k^1 = M_{k+1}$ and $M_k^{\ell+1} \subset \mathrm{int}(M_k^\ell)$ for all $\ell \in \mathbb{N}$. This is possible since M_k is contained in the interior of M_{k+1}. Then we have $\partial M \cap \partial M_k^\ell \subset \partial M_{k+1}$ and ∂M_k^ℓ coincides with ∂M_{k+1} near every point of this intersection, hence for all $\ell \in \mathbb{N}$

$$*\alpha_k\big|_{\partial M_k^\ell} = 0 \quad \text{on } \partial M \cap \partial M_k^\ell.$$

Thus one can iterate corollary 9.6 (ii) to deduce that $u_k^*A|_{M_k}$ is smooth: We will prove by induction that $\alpha_k|_{M_k^\ell} \in W^{\ell,p}(M_k^\ell, \mathrm{T}^*M_k^\ell \otimes \mathfrak{g}_P)$ for all $\ell \in \mathbb{N}$ (and this proves that $u_k^*A|_{M_k}$ is smooth since $M_k \subset M_k^\ell$ for all $\ell \in \mathbb{N}$) :

Assume that $\alpha_k|_{M_k^\ell} \in W^{\ell,p}(M_k^\ell, \mathrm{T}^*M_k^\ell \otimes \mathfrak{g}_P)$ for some $\ell \in \mathbb{N}$ (which is true for $\ell = 1$). Then corollary 9.6 (ii) with $M' = M_k^\ell$ and $M'' = M_{k+1}^\ell$ implies $\alpha_k|_{M_k^{\ell+1}} \in W^{\ell+1,q}(M_k^{\ell+1}, \mathrm{T}^*M_k^{\ell+1} \otimes \mathfrak{g}_P)$. In case $\ell \geq 2$ or $p > n$ we have $q = p$, so this proves the iteration step. In case $\ell = 1$ and $p < n$ this requires a further iteration (for $p = n$ one starts with a smaller $\frac{n}{2} < p < n$, then one still has $W^{1,p}$-regularity on the compact manifold M_{k+1}):

Choose a sequence of compact submanifolds $M_k^2 \subset N_i \subset M_{k+1}$ such that $N_{-1} = M_{k+1}$ and N_i is contained in the interior of N_{i-1} for all $i \in \mathbb{N}_0$. We then iterate corollary 9.6 (ii) with $M' = N_i$ and $M'' = N_{i+1}$. (Note that the boundary condition is satisfied as before.) This yields $\alpha_k|_{N_i} \in W^{2,q_i}(N_i, \mathrm{T}^*N_i \otimes \mathfrak{g}_P)$ with the sequence q_i as in lemma 9.7. Again this sequence arrives at some $q_j \geq p$ and hence $\alpha_k|_{M_k^2} \in W^{2,p}(M_k^2, \mathrm{T}^*M_k^2 \otimes \mathfrak{g}_P)$.

So for every $k \in \mathbb{N}$ this proves that $\alpha_k|_{M_k}$ and hence $u_k^*A|_{M_k} = (\tilde{A}_k + \alpha_k)|_{M_k}$ is smooth. Now apply proposition 9.8 to the gauge transformations $u_k \in \mathcal{G}^{2,p}(P|_{M_k})$ to obtain a gauge transformation $u \in \mathcal{G}^{2,p}_{loc}(P)$ such that u^*A is smooth. $\square$

Chapter 10

Proof of Strong Compactness

In this chapter we prove the strong Uhlenbeck compactness theorem E and its generalization, theorem E′, to noncompact manifolds.

So we consider a principal G-bundle $P \to M$ over a Riemannian n-manifold M, where G is a compact Lie group. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$, and in case $n = 2$ assume in addition $p > \frac{4}{3}$.[1] Now the strong Uhlenbeck compactness theorem E for compact base manifolds with boundary uses the weak Yang-Mills equation (9.2) and can be restated as follows.

Theorem 10.1 (Strong Uhlenbeck Compactness)
Assume M is compact. Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}(P)$ be a sequence of weak Yang-Mills connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^{\nu\,*}A^\nu$ converges uniformly with all derivatives to a smooth connection $\tilde{A} \in \mathcal{A}(P)$.

The idea by Salamon for this proof is as follows: Firstly, by the weak Uhlenbeck compactness one finds a subsequence and a sequence of gauge transformations such that the transformed connections converge $W^{1,p}$-weakly. The limit connection also is a weak Yang-Mills connection.[2] So after a gauge transformation a subsequence of the connections is $W^{1,p}$-bounded and converges to a smooth connection in the L^q-norm for q such that the Sobolev embedding $W^{1,p} \hookrightarrow L^q$ is compact. One then puts the connections into relative Coulomb gauge with respect to this smooth limit connection. Now the Yang-Mills equation together with the relative Coulomb

[1] In case $n = 2$ the weak Yang-Mills equation (9.2) is only well posed under the assumption $p \geq \frac{4}{3}$, see definition 9.1. The strict inequality $p > \frac{4}{3}$ becomes necessary if one wants to use the local slice theorem 8.1 for the proof.

[2] This requires $p > \frac{4}{3}$ in case $n = 2$ for the strong L^{p^*}-convergence of the connections.

gauge provide uniform bounds on all $W^{k,p}$-norms of the connections, and the compactness then follows from compact Sobolev embeddings.

The usual approach to the proof of theorem 10.1 is the same as in the proof of the weak Uhlenbeck compactness theorem 7.1. One uses the L^p-bound on the curvature to find local trivializations in which the given sequence of connections can be put into Uhlenbeck gauge. In particular, one has $W^{1,p}$-bounds on the connections in that gauge. If the connections moreover satisfy the weak Yang-Mills equation, then one obtains $W^{k,p}$-bounds for all $k \in \mathbb{N}$ from proposition 9.5. Here it is essential to have the local Uhlenbeck gauge with respect to the given metric that also enters in the weak Yang-Mills equation. This is why we proved theorem B for a general metric, not just the Euclidean metric on balls (cf. theorem 6.3). Now the same patching construction as for the weak compactness, lemma 7.2, can be used to patch the local gauge transformations into global ones such that there are global $W^{k,p}$-bounds on a subsequence of the gauge transformed connections. Then the compactness follows from compact Sobolev embeddings. In fact, one can simply follow through the proof of the weak compactness theorem 7.1, just replacing $W^{1,p}$-bounds by $W^{k,p}$-bounds for all $k \in \mathbb{N}$.

Note that this argument proves theorem 10.1 also in case $n = 2, p = \frac{4}{3}$ since one does not use the fact that the weak $W^{1,p}$-limit of a sequence of weak Yang-Mills connections also satisfies the weak Yang-Mills equation. However, this argument is limited in another respect. It does not exhibit so clearly that the analogue of the strong Uhlenbeck compactness theorem is true for all equations in gauge theory that form an elliptic system together with the relative Coulomb gauge conditions. In particular, if one considers nonlocal boundary conditions as in [W], [3] then compactness cannot be proven in local trivializations.

The main lemma for our proof of theorem 10.1 is the following consequence of the local slice theorem 8.1 together with an iteration of the regularity result for Yang-Mills connections in proposition 9.5 (i).

Lemma 10.2 *Assume M is compact, let $\tilde{A} \in \mathcal{A}(P)$ be a smooth reference connection, fix a constant c, and let $p \leq q < \infty$ be such that*

$$\tfrac{1}{n} > \tfrac{1}{q} > \tfrac{1}{p} - \tfrac{1}{n}.$$

Then there exist constants $(C_\ell)_{\ell \in \mathbb{N}_0}$ and $\delta > 0$ such that the following holds: For every weak Yang-Mills connection $A \in \mathcal{A}^{1,p}(P)$ that satisfies

$$\|A - \tilde{A}\|_q \leq \delta, \qquad \|A - \tilde{A}\|_{W^{1,p}} \leq c \tag{10.1}$$

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(P)$ such that

$$\|u^*A - \tilde{A}\|_q \leq C_0 \|A - \tilde{A}\|_q, \qquad \|u^*A - \tilde{A}\|_{W^{\ell,p}} \leq C_\ell \qquad \forall \ell \in \mathbb{N}.$$

[3] There the boundary of the base manifold M is $\mathbb{R} \times \Sigma$ with a Riemann surface Σ. A Lagrangian in the space of flat connections over Σ is fixed. Then one considers anti-self-dual instantons that on each boundary slice $\{s\} \times \Sigma$ restrict to a connection in that Lagrangian. This is an elliptic equation in the relative Coulomb gauge and so one obtains the corresponding compactness result.

Proof: Fix a constant c and let $\delta > 0$ be the constant from theorem 8.1 with $c_0 = c$ and the reference connection $\tilde{A}$. Now consider any connection $A \in \mathcal{A}^{1,p}(P)$ that satisfies (10.1). By the local slice theorem 8.1 one finds a gauge transformation $u \in \mathcal{G}^{2,p}(P)$ that puts A into relative Coulomb gauge with respect to $\tilde{A}$. This directly implies the inequality on $\|u^*A - \tilde{A}\|_q$ with $C_0 = C_{CG}$, so it remains to inductively find the constants C_ℓ. The first constant also is provided by a relative Coulomb gauge condition:

$$\|u^*A - \tilde{A}\|_{W^{1,p}} \;\leq\; C_{CG}\|A - \tilde{A}\|_{W^{1,p}} \;\leq\; C_{CG}c \;=:\; C_1. \tag{10.2}$$

The relative Coulomb gauge moreover asserts that $u^*A = \tilde{A} + \alpha$ solves the first differential equation in corollary 9.6 (i):

$$\begin{cases} \mathrm{d}^*_{\tilde{A}}\alpha = 0, \\ *\alpha|_{\partial M} = 0. \end{cases}$$

Now if A is moreover a weak Yang-Mills connection, then lemma 9.2 asserts that u^*A also solves the weak Yang-Mills equation

$$\int_M \langle\, F_{u^*A} \,,\, \mathrm{d}_{u^*A}\beta \,\rangle = 0 \qquad \forall \beta \in \Omega^1(M; \mathfrak{g}_P).$$

Now corollary 9.6 (i) can be iterated to provide the constants C_ℓ. The first step gives $u^*A \in \mathcal{A}^{2,q}(P)$, where $q \leq p$, and with finite constants C, C'

$$\begin{aligned} \|u^*A - \tilde{A}\|_{W^{2,q}} \;&=\; \|\alpha\|_{W^{2,q}} \;\leq\; C\left(1 + \|\alpha\|_{W^{1,p}} + \|\alpha\|^3_{W^{1,p}}\right) \\ &\leq\; C'(1 + C_1 + C_1^3) \;=:\; \tilde{C}_0. \end{aligned}$$

In case $p > n$ we have $q = p$, so this gives the uniform bound $C_2 := \tilde{C}_0$. In case $p < n$ we have $q = \frac{np}{2n-p} =: q_0$, so for $i = 0$ this gives $u^*A \in \mathcal{A}^{2,q_i}(P)$ and

$$\|u^*A - \tilde{A}\|_{W^{2,q_i}} \leq \tilde{C}_i. \tag{10.3}$$

By an iteration analogous to the proof of theorem 9.4 we find constants such that this in fact holds for the sequence q_i as in lemma 9.7: If (10.3) holds for some $q_i < p$, then the Sobolev embedding $W^{2,q_i} \hookrightarrow W^{1,p_i}$ together with corollary 9.6 (i) implies $u^*A \in \mathcal{A}^{2,q_{i+1}}(P)$ and for some constants C, C'

$$\begin{aligned} \|u^*A - \tilde{A}\|_{W^{2,q_{i+1}}} \;&=\; \|\alpha\|_{W^{2,q_{i+1}}} \;\leq\; C\left(1 + \|\alpha\|_{W^{1,p_i}} + \|\alpha\|^3_{W^{1,p_i}}\right) \\ &\leq\; C'\left(1 + \|\alpha\|_{W^{2,q_i}} + \|\alpha\|^3_{W^{2,q_i}}\right) \\ &\leq\; C'(1 + \tilde{C}_i + \tilde{C}_i^3) \;=:\; \tilde{C}_{i+1}. \end{aligned}$$

Lemma 9.7 asserts that $q_j \geq p$ for some finite $j \in \mathbb{N}$, so this yields $C_2 = \tilde{C}_j$ (possibly corrected by another constant for the embedding $W^{2,q_j} \hookrightarrow W^{2,p}$). In case $p = n$ we choose a smaller $\frac{n}{2} < p' < n$. For this (10.2) also holds with another constant and above iteration yields $u^*A \in \mathcal{A}^{2,p'}(P)$ with a uniform bound $\|u^*A - \tilde{A}\|_{W^{2,p'}} \leq C'$. Now one has the Sobolev embedding $W^{2,p'} \hookrightarrow W^{1,2p'}$ and the corollary yields the $W^{2,2p'}$-regularity and -bound since $2p' > n$. So by the bounded embedding $W^{2,2p'} \hookrightarrow W^{2,n}$ we also find a uniform bound C_2 in the case $p = n$.

The further bounds C_ℓ are now established by a straight iteration of corollary 9.6 (i): Suppose that $u^*A \in \mathcal{A}^{\ell,p}(P)$ and the constants are found up to C_ℓ, then the corollary asserts that $u^*A \in \mathcal{A}^{\ell+1,p}(P)$ and

$$\begin{aligned}\|u^*A - \tilde{A}\|_{W^{\ell+1,p}} &\leq C\left(1 + \|\alpha\|_{W^{\ell,p}} + \|\alpha\|^3_{W^{\ell,p}}\right) \\ &\leq C(1 + C_\ell + C_\ell^3) \;=:\; C_{\ell+1}.\end{aligned}$$

□

Proof of theorem 10.1 :
The weak compactness theorem 7.1 provides a subsequence (which we again denote by $(A^\nu)_{\nu\in\mathbb{N}}$) and gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ such that $u^{\nu\,*}A^\nu$ converges in the weak $W^{1,p}$-topology to some $A \in \mathcal{A}^{1,p}(P)$ and $\left\|u^{\nu\,*}A^\nu - A\right\|_{W^{1,p}}$ is bounded. (The boundedness follows from the weak convergence by e.g. [Y, V.1,Thm.3])

Now let $q := \sup\{2p, p^*\}$, where $\frac{1}{p^*} = 1 - \frac{1}{p}$. Then q satisfies the assumptions of lemma 10.2. Indeed, $q \geq 2p > n$ and $\frac{1}{2p} > \frac{1}{p} - \frac{1}{n}$ due to $p > \frac{n}{2}$. Furthermore, $\frac{1}{p^*} > \frac{1}{p} - \frac{1}{n}$ is equivalent to $p > \frac{2n}{n+1}$; for $n = 1$ this is met due to $p > 1$, for $n = 2$ this requires $p > \frac{4}{3}$, and for $n \geq 3$ this holds by $p > \frac{n}{2}$.

The condition on q implies that the embedding $W^{1,p} \hookrightarrow L^q$ is compact, hence a subsequence of the $u^{\nu\,*}A^\nu$ also converges in the L^q-norm to A. Again denote that subsequence by $(A^\nu)_{\nu\in\mathbb{N}}$. This sequence in $\mathcal{A}^{1,p}(P)$ converges to $A \in \mathcal{A}^{1,p}(P)$ in the L^q-norm and in the weak $W^{1,p}$-topology.

So far we have only used the L^p-bound on the curvature. Now moreover, the A^ν are weak Yang-Mills connections (the weak Yang-Mills equation is invariant under gauge transformations by lemma 9.2). Hence the limit connection A also solves the weak Yang-Mills equation: For all $\beta \in \Omega^1(M;\mathfrak{g}_P)$

$$\int_M \langle\, F_A\, ,\, \mathrm{d}_A\beta\,\rangle \;=\; \lim_{\nu\to\infty} \int_M \langle\, F_{A^\nu}\, ,\, \mathrm{d}_{A^\nu}\beta\,\rangle \;=\; 0$$

Indeed, $\mathrm{d}_{A^\nu}\beta$ converges in the L^{p^*}-norm to $\mathrm{d}_A\beta$ since $q \geq p^*$, and F_{A^ν} converges in the weak $W^{1,p}$-topology to F_A. The latter follows from the weak L^p-convergence of the local representatives $\left(F_{A^\nu}\right)_\alpha = \mathrm{d}A^\nu_\alpha + [A^\nu_\alpha \wedge A^\nu_\alpha]$ in all bundle charts. The second term even converges strongly since the A^ν_α converge in the L^{2p}-norm due to $q \geq 2p$. For the weak convergence of $\mathrm{d}A^\nu_\alpha$ first only test with smooth $\beta \in \Omega^2(U_\alpha;\mathfrak{g})$

that vanish on ∂U_α. For these

$$\int_{U_\alpha} \langle \mathrm{d}A^\nu_\alpha , \beta \rangle = \int_{U_\alpha} \langle A^\nu_\alpha , \mathrm{d}^*\beta \rangle \underset{\nu\to\infty}{\longrightarrow} \int_{U_\alpha} \langle A_\alpha , \mathrm{d}^*\beta \rangle = \int_{U_\alpha} \langle \mathrm{d}A_\alpha , \beta \rangle .$$

Then the limit $\int_{U_\alpha} \langle \mathrm{d}A^\nu_\alpha , \beta \rangle \longrightarrow \int_{U_\alpha} \langle \mathrm{d}A_\alpha , \beta \rangle$ as $\nu \to \infty$ in fact holds for all smooth $\beta \in \Omega^2(U_\alpha; \mathfrak{g})$ since these can be L^{p^*}-approximated by such forms that vanish on the boundary, and since the $\mathrm{d}A^\nu$ are L^p-bounded. So we have seen that A is a weak Yang-Mills connection. (Here it was crucial to have the compact Sobolev embedding $W^{1,p} \hookrightarrow L^{p^*}$, which requires $p > \frac{4}{3}$ in case $n = 2$.)

Next, theorem 9.4 (i) provides a gauge transformation $\tilde{u} \in \mathcal{G}^{2,p}(P)$ such that $\tilde{A} := \tilde{u}^*A$ is smooth. Now the $\tilde{A}^\nu := \tilde{u}^*A^\nu$ converge to $\tilde{A}$ in the L^q-norm and satisfy a uniform bound $\|\tilde{A}^\nu - \tilde{A}\|_{W^{1,p}} \leq c$ for some constant c. This is due to the continuity of the gauge action in lemma A.6. For the L^q-convergence use that lemma with p replaced by q and note that we have the Sobolev embeddings $\mathcal{G}^{2,p}(P) \subset \mathcal{G}^{1,q}(P)$ and $\mathcal{A}^{1,p}(P) \subset \mathcal{A}^{0,q}(P)$. In order to prove the theorem we then have to find gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ such that a subsequence of the $u^{\nu\,*}\tilde{A}^\nu$ converges in the uniform $\mathcal{C}^\infty$-topology. (Recall that $\mathcal{G}^{2,p}(P)$ is closed under composition by lemma A.5.)

Let $\delta > 0$ be determined by $\tilde{A}$, q, and c as in lemma 10.2. Then there is $\nu_0 \in \mathbb{N}$ such that $\|\tilde{A}^\nu - \tilde{A}\|_q \leq \delta$ for all $\nu \geq \nu_0$, and hence lemma 10.2 applies to $\tilde{A}^\nu$. Thus one finds gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ for all $\nu \geq \nu_0$ such that

$$\|u^{\nu\,*}\tilde{A}^\nu - \tilde{A}\|_q \leq C_0 \|\tilde{A}^\nu - \tilde{A}\|_q .$$

So the $u^{\nu\,*}\tilde{A}^\nu$ converge to $\tilde{A}$ in the L^q-norm. Moreover, there are uniform bounds for all $\ell \in \mathbb{N}$,

$$\|u^{\nu\,*}\tilde{A}^\nu - \tilde{A}\|_{W^{\ell,p}} \leq C_\ell \qquad \forall \nu \geq \nu_0 .$$

For every $\ell \in \mathbb{N}$ there is a compact Sobolev embedding $W^{\ell+2,p} \hookrightarrow \mathcal{C}^\ell$. Thus for all $\ell \in \mathbb{N}$ we find a further subsequence of the $u^{\nu\,*}\tilde{A}^\nu$ that converges in the uniform $\mathcal{C}^\ell$-topology. So by fixing one further element of the sequence in every step we obtain a sequence that converges in the uniform $\mathcal{C}^\infty$-topology. The limit has to be $\tilde{A}$ since this already was the L^q-limit. □

The Yang-Mills equation with boundary condition can also be used to generalize the strong Uhlenbeck compactness to noncompact base manifolds (with possibly nonempty boundary) that are exhausted by compact deformation retracts. Recall that we only consider increasing exhausting sequences $M = \bigcup_{k\in\mathbb{N}} M_k$ such that each M_k lies in the interior of M_{k+1}. In the following theorem we additionally consider a perturbation of the Yang-Mills equation by a sequence of $\mathcal{C}^\infty$-convergent metrics on M. This generalization of strong Uhlenbeck compactness also holds for compact base manifolds M (just set $M_k := M$), but we preferred to keep theorem 10.1 as simple as possible and put all the technicalities into this theorem E′.

Theorem 10.3 *Assume $M = \bigcup_{k\in\mathbb{N}} M_k$ is exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Let $(g_\nu)_{\nu\in\mathbb{N}}$ be a sequence of metrics on M that converges uniformly with all derivatives on every compact set. For all $\nu\in\mathbb{N}$ let $A^\nu \in \mathcal{A}^{1,p}_{loc}(P)$ be a weak Yang-Mills connection with respect to g_ν and suppose that for all $k\in\mathbb{N}$*

$$\sup_{\nu\in\mathbb{N}} \|F_{A^\nu}\|_{L^p(M_k)} < \infty.$$

Then there exists a subsequence (again denoted $(A^\nu)_{\nu\in\mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^{\nu\,*}A^\nu$ converges uniformly with all derivatives on every compact set to a smooth connection $\tilde{A}\in\mathcal{A}(P)$.

The limit of the metrics g_ν is a smooth metric g on M. The above L^p-norm can be defined with respect to that metric or with respect to the g_ν – the norms are all equivalent. In the following, however, we will have to distinguish the metrics: If a norm or an operator is not calculated with respect to the metric g, then the other metric will be indicated.

Similarly to the weak Uhlenbeck compactness one uses proposition 7.6 to generalize the strong Uhlenbeck compactness on compact manifolds to this result for manifolds that are exhausted by compact deformation retracts. Note however that one cannot simply apply the strong Uhlenbeck compactness theorem 10.1 to the given sequence of connections restricted to the compact manifolds M_k – due to the boundary condition the given connections do not restrict to Yang-Mills connections on the subsets of M. So in order to obtain the uniform bounds assumed in proposition 7.6 we establish the following analogon of lemma 10.2. This also takes care of the varying metrics in the Yang-Mills equation.

Lemma 10.4 *Let $M'' \subset M' \subset M$ be compact submanifolds such that M'' is contained in the interior of M'. Fix a metric g on M and a smooth reference connection $\tilde{A}\in\mathcal{A}(P)$, and let $p\le q<\infty$ be such that*

$$\tfrac{1}{n} > \tfrac{1}{q} > \tfrac{1}{p} - \tfrac{1}{n}.$$

Then for every constant c there exist constants $\delta,\varepsilon>0$, $(C_\ell)_{\ell\in\mathbb{N}}$, and $(\varepsilon_\ell)_{\ell\in\mathbb{N}}>0$ such that the following holds: Let g' be a metric on M with

$$\|g'-g\|_{W^{1,\infty}(M')} \le \varepsilon. \tag{10.4}$$

Suppose that $A\in\mathcal{A}^{1,p}_{loc}(P)$ is a weak Yang-Mills connection with respect to the metric g' and satisfies

$$\|A-\tilde{A}\|_{L^q(M')} \le \delta, \qquad \|A-\tilde{A}\|_{W^{1,p}(M')} \le c. \tag{10.5}$$

Then there exists a gauge transformation $u\in\mathcal{G}^{2,p}(P|_{M'})$ such that $u^*A|_{M''}$ is smooth. Moreover, for all $\ell\in\mathbb{N}$ such that $\|g'-g\|_{W^{\ell,\infty}(M')}\le\varepsilon_\ell$ one has

$$\|u^*A-\tilde{A}\|_{W^{\ell,p}(M'')} \le C_\ell.$$

Proof: This is proven analogous to lemma 10.2, just that instead of (i) in corollary 9.6 we have to iterate (ii) and take care of the sequence of metrics. This iteration in turn is similar to the proof of theorem 9 (ii); we only have to put in the according estimates, which is slightly subtle due to the varying metrics.

Let a constant c and a smooth reference connection $\tilde{A}$ be given. Then let $\varepsilon, \delta > 0$ be the constants from theorem 8.1 and remark 8.2 (iii) on M' with $c_0 = c$. Fix a metric g' on M that satisfies (10.4), and let $A \in \mathcal{A}^{1,p}_{loc}(P)$ be a weak Yang-Mills connection with respect to g' that moreover satisfies (10.5). Then by theorem 8.1 there exists a gauge transformation $u \in \mathcal{G}^{2,p}(P|_{M'})$ such that $u^*A|_{M'} \in \mathcal{A}^{1,p}(P|_{M'})$ is in relative Coulomb gauge with respect to $\tilde{A}$ and the metric g'. This directly provides the first constant,

$$\begin{aligned} \|u^*A - \tilde{A}\|_{W^{1,p}(M'')} &\leq \|u^*A - \tilde{A}\|_{W^{1,p}(M')} \\ &\leq C_{CG}\|A - \tilde{A}\|_{W^{1,p}(M')} \;\leq\; C_{CG}c \;=:\; C_1 \end{aligned} \tag{10.6}$$

Here the constant C_{CG} from theorem 8.1 is independent of the metric g', hence so is C_1. This did not require a further bound on the metric, so we can set $\varepsilon_1 := \varepsilon$. The constants ε_ℓ and C_ℓ for $\ell \geq 2$ are then inductively found using corollary 9.6 (ii):

Fix a sequence of compact submanifolds $M'' \subset N_\ell \subset M'$ such that $N_1 = M'$ and $N_{\ell+1}$ lies in the interior of N_ℓ. This exists since M'' is contained in the interior of M'. Now the relative Coulomb gauge conditions $\mathrm{d}^*_{\tilde{A}}(u^*A - \tilde{A})|_{M'} = 0$ and $*(u^*A - \tilde{A})|_{\partial M'} = 0$ assert that $\alpha := u^*A - \tilde{A}$ satisfies for all $\ell \in \mathbb{N}$ the following with respect to the metric g' :

$$\begin{cases} \mathrm{d}^*_{\tilde{A}}\alpha = 0 & \text{on } N_\ell \\ *\alpha|_{\partial N_\ell} = 0 & \text{on } \partial M \cap \partial N_\ell. \end{cases}$$

Moreover, A is a weak Yang-Mills connection on M with respect to g', so lemma 9.2 assserts for all smooth 1-forms $\beta \in \Omega^1(M;\mathfrak{g}_P)$ supported in N_ℓ

$$\int_M \langle\, F_{u^*A}\,,\, \mathrm{d}_{u^*A}\,\beta\,\rangle_{g'} = 0.$$

Thus for all $\ell \in \mathbb{N}$ the connection $u^*A|_{N_\ell} \in \mathcal{A}^{1,p}(P|_{N_\ell})$ satisfies the weak and strong equations in corollary 9.6 (ii) on N_ℓ for the metric g'. So the corollary can be iterated to prove $u^*A|_{N_\ell} \in \mathcal{A}^{\ell,p}(P|_{N_\ell})$ and provide constants $\varepsilon_\ell > 0$ and C_ℓ on the sucessively smaller domains $N_\ell \supset M''$ such that $u^*A|_{N_\ell} \in \mathcal{A}^{\ell,p}(P|_{N_\ell})$ and the following implication holds:

$$\|g' - g\|_{W^{\ell,\infty}(M')} \leq \varepsilon_\ell \quad\Longrightarrow\quad \|u^*A - \tilde{A}\|_{W^{\ell,p}(N_\ell)} \leq C_\ell. \tag{10.7}$$

For $\ell = 1$ with $N_1 = M'$ this was established in (10.6). Now suppose this is true for some $\ell \in \mathbb{N}$. Then corollary 9.6 (ii) gives $u^*A|_{N_{\ell+1}} \in \mathcal{A}^{\ell+1,q}(P|_{N_{\ell+1}})$ and

$$\begin{aligned} \|u^*A - \tilde{A}\|_{W^{\ell+1,q}(N_{\ell+1})} &\leq C\big(1 + \|u^*A - \tilde{A}\|_{W^{\ell,p}(N_\ell)} + \|u^*A - \tilde{A}\|^3_{W^{\ell,p}(N_\ell)}\big) \\ &\leq C(1 + C_\ell + C_\ell^3) \;=:\; C_{\ell+1}. \end{aligned}$$

For this estimate we have to require $\|g'-g\|_{W^{\ell+1,\infty}(M')} \leq \varepsilon_{\ell+1}$ for some sufficiently small $\varepsilon_\ell \geq \varepsilon_{\ell+1} > 0$ in order to have the bound C_ℓ and so that we can choose the above constant C from corollary 9.6 (ii) to be independent of the metric g'. (The constant depends continuously on the metric with respect to the $W^{\ell+1,\infty}$-topology.) In case $\ell \geq 2$ or $p > n$ we have $q = p$, so this gives the required constants $C_{\ell+1}$ and $\varepsilon_{\ell+1}$. In case $\ell = 1$ and $p \leq n$ we need a separate iteration of the corollary to find ε_2 and C_2 for (10.7):

Firstly, if $p = n$ then replace it by some $\frac{n}{2} < p' < n$ and start the iteration from (10.7) for $\ell = 1$ with p replaced by p'. (Recall that the Sobolev norms are all defined with respect to the metric g, so the constant in the estimate between the $W^{1,p'}$- and $W^{1,p}$-norm is independent of the metric g'.) Now fix another sequence of compact submanifolds $N_2 \subset \tilde{N}_i \subset N_1 = M'$ such that $\tilde{N}_{-1} = M'$ and $\tilde{N}_{i+1}$ lies in the interior of $\tilde{N}_i$. Then the use of corollary 9.6 (ii) as above on $\tilde{N}_0$ instead of N_2 with $q = q_0 = \frac{np}{2n-p}$ gives the following for $i = 0$: The regularity $u^*A|_{\tilde{N}_i} \in \mathcal{A}^{2,q_i}(P|_{\tilde{N}_i})$ holds and there are constants $\tilde{\varepsilon}_i > 0$ and $\tilde{C}_i$ such that

$$\|g' - g\|_{W^{2,\infty}(M')} \leq \tilde{\varepsilon}_i \quad \Longrightarrow \quad \|u^*A - \tilde{A}\|_{W^{2,q_i}(\tilde{N}_i)} \leq \tilde{C}_i. \tag{10.8}$$

Suppose this holds for $i \in \mathbb{N}_0$, then the Sobolev estimate for $W^{2,q_i} \hookrightarrow W^{1,p_i}$ on $\tilde{N}_i$ (with q_i, p_i as in lemma 9.7) implies $u^*A|_{\tilde{N}_i} \in \mathcal{A}^{1,p_i}(P|_{\tilde{N}_i})$ and

$$\|g' - g\|_{W^{2,\infty}(M')} \leq \tilde{\varepsilon}_i \quad \Longrightarrow \quad \|u^*A - \tilde{A}\|_{W^{1,p_i}(\tilde{N}_i)} \leq C_S\tilde{C}_i. \tag{10.9}$$

Here the Sobolev constant C_S is independent of the metric g'. Now apply corollary 9.6 (ii) to obtain (10.8) for $i+1$: This yields $u^*A|_{\tilde{N}_{i+1}} \in \mathcal{A}^{1,p_{i+1}}(P|_{\tilde{N}_{i+1}})$, and if $\|g'-g\|_{W^{2,\infty}(M')} \leq \tilde{\varepsilon}_{i+1}$ for sufficiently small $\tilde{\varepsilon}_i \geq \tilde{\varepsilon}_{i+1} > 0$ (such that the constant C below is independent of the metric g') then

$$\begin{aligned}\|u^*A - \tilde{A}\|_{W^{2,q_{i+1}}(\tilde{N}_{i+1})} &\leq C\big(1 + \|u^*A - \tilde{A}\|_{W^{1,p_i}(\tilde{N}_i)} + \|u^*A - \tilde{A}\|^3_{W^{1,p_i}(\tilde{N}_i)}\big) \\ &\leq C(1 + C_S\tilde{C}_i + (C_S\tilde{C}_i)^3) \;=:\; \tilde{C}_{i+1}.\end{aligned}$$

Thus we have proven (10.8) for the whole sequence q_i, which by lemma 9.7 terminates at some $q_j \geq p$. Now choose $\tilde{\varepsilon}_j \geq \varepsilon_2 > 0$ sufficiently small such that there is a uniform constant C in the estimate for $W^{2,q_j} \hookrightarrow W^{2,p}$ on $\tilde{N}_j$, then we have established (10.7) for $\ell = 2$ with $C_\ell := C\tilde{C}_j$. This was the missing induction step in case $p < n$.

In case $p = n$ we have proven (10.8) for $q_j \geq p' > \frac{n}{2}$ and thus obtain (10.9) with $p_j := 2p'$ from the Sobolev estimate for $W^{2,p'} \hookrightarrow W^{1,2p'}$. Now corollary 9.6 (ii) on $\tilde{N}_j \subset N_2$ together with the estimate for $W^{2,2p'} \hookrightarrow W^{2,n}$ provides constants $\varepsilon_2 > 0$ and C_2 such that (10.7) holds for $\ell = 2$.

Thus we have inductively defined the constants $\varepsilon_\ell > 0$ and C_ℓ for all cases and proved $u^*A|_{N_\ell} \in \mathcal{A}^{\ell,p}(P|_{N_\ell})$ and (10.7) for all $\ell \in \mathbb{N}$. This proves the lemma since $M'' \subset N_\ell$ for all $\ell \in \mathbb{N}$. □

Proof of theorem 10.3 :
The weak compactness theorem 7.5 provides a subsequence (again denoted by $(A^\nu)_{\nu\in\mathbb{N}}$) and gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that the connections $u^{\nu\,*}A^\nu$ converge in the weak $W^{1,p}$-topology when restricted to any of the submanifolds M_k. Thus there exists a global connection $A \in \mathcal{A}^{1,p}_{loc}(P)$ such that $A|_{M_k} \in \mathcal{A}^{1,p}(P|_{M_k})$ is the limit of $u^{i\,*}A^{\nu_i}|_{M_k}$ for all $k \in \mathbb{N}$. This also implies that for all $k \in \mathbb{N}$ there is a uniform bound on $\|u^{\nu\,*}A^\nu - A\|_{W^{1,p}(M_k)}$. (This uniform bound follows from the weak convergence, see e.g. [Y, V.1,Thm.3].)

Define $q := \sup\{2p, p^*\}$ as for theorem 10.1 such that theorem 8.1 can be applied and the embedding $W^{1,p} \hookrightarrow L^q$ is compact. Then we find a (diagonal) subsequence (again labelled by $\nu \in \mathbb{N}$) of the $u^{\nu\,*}A^\nu$ that on every M_k converges in the L^q-norm to $A|_{M_k}$.

As in the compact case the $u^{\nu\,*}A^\nu$ and the limit connection A are weak Yang-Mills connections. (One only has to test (9.2) with $\beta \in \Omega^1(M;\mathfrak{g}_P)$ that have compact support hence are supported in some M_k, then the argument for theorem 10.1 applies.) Hence theorem 9.4 (ii) provides a gauge transformation $\tilde{u} \in \mathcal{G}^{2,p}_{loc}(P)$ such that $\tilde{A} := \tilde{u}^*A$ is smooth and moreover $\tilde{u}|_{M_k} \in \mathcal{G}^{2,p}(P|_{M_k})$ for all $k \in \mathbb{N}$. Due to that regularity the new connections $\tilde{A}^\nu := \tilde{u}^*u^{\nu\,*}A^\nu = (u^\nu\tilde{u})^*A^\nu$ converge to $\tilde{A}$ in the L^q-norm on every M_k and satisfy uniform bounds $\|\tilde{A}^\nu - \tilde{A}\|_{W^{1,p}(M_k)} \le c_k$ for all $k \in \mathbb{N}$ and some constants c_k. This follows as before from the continuity of the gauge action in lemma A.6. Now the task is to find gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that a subsequence of the $u^{\nu\,*}\tilde{A}^\nu$ converges in the uniform $\mathcal{C}^\infty$-topology on all compact subsets.

This strong compactness will be a consequence of the uniform $W^{\ell,p}$-bounds from proposition 7.6 for all $\ell \in \mathbb{N}$. So we have to verify the assumptions of proposition 7.6 with $I := \mathbb{N}$ for the sequence $(\tilde{A}^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{1,p}_{loc}(P)$ of connections: Fix $k \in \mathbb{N}$ and consider a subsequence of these connections – again denoted by $(\tilde{A}^\nu)_{\nu\in\mathbb{N}}$. Then one has to find a subsequence $(\nu_{k,i})_{i\in\mathbb{N}}$ and gauge transformations $u^{k,i} \in \mathcal{G}^{2,p}(P|_{M_k})$ such that [4]

$$\sup_{i\in\mathbb{N}} \left\| u^{k,i\,*}\tilde{A}^{\nu_{k,i}} - \tilde{A} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall \ell \in \mathbb{N}.$$

For that purpose apply lemma 10.4 with $M'' := M_k$, $M' := M_{k+1}$, and $c := c_{k+1}$. The metric g is given by the limit of the g_ν, and the reference connection is $\tilde{A}$. Let $\delta, \varepsilon > 0$, $(C_\ell)_{\ell\in\mathbb{N}}$, and $(\varepsilon_\ell)_{\ell\in\mathbb{N}} > 0$ be the constants given by that lemma.

[4]In chapter 7 there is a fixed smooth reference connection $\tilde{A}$ with respect to which all Sobolev norms on the affine space of connections are defined. So $\|A\|_{W^{\ell,p}}$ in the notation of proposition 7.6 is identical to $\|A - \tilde{A}\|_{W^{\ell,p}}$ in the notation of this chapter.

Recall that the metrics g_ν converge to g uniformly with all derivatives, and the connections $\tilde{A}^\nu$ converge to $\tilde{A}$ in the L^q-norm on M_{k+1}. So one finds an $N \in \mathbb{N}$ such that for all $\nu \geq N$

$$\|g^\nu - g\|_{W^{1,\infty}(M_{k+1})} \leq \varepsilon,$$

$$\|\tilde{A}^\nu - \tilde{A}\|_{L^q(M_{k+1})} \leq \delta, \qquad \|\tilde{A}^\nu - \tilde{A}\|_{W^{1,p}(M_{k+1})} \leq c = c_{k+1}.$$

These are the assumptions of lemma 10.4 for the metric g_ν and the weak Yang-Mills connection $\tilde{A}^\nu$ (with respect to g_ν). Thus the lemma provides gauge transformations $u^\nu \in \mathcal{G}^{2,p}(M_{k+1})$ for all $\nu \geq N$ such that $u^{\nu\,*}\tilde{A}^\nu|_{M_k}$ is smooth. Moreover, due to the convergence of the metrics g_ν one finds a sequence $(N_\ell)_{\ell\in\mathbb{N}}$ of integers $N_\ell > N$ such that $\|g - g_\nu\|_{W^{\ell,\infty}(M_{k+1})} \leq \varepsilon_\ell$ holds for all $\ell \in \mathbb{N}$ and $\nu \geq N_\ell$. So lemma 10.4 furthermore asserts that for all $\ell \in \mathbb{N}$

$$\sup_{\nu \geq N_\ell} \left\| u^{\nu\,*}\tilde{A}^\nu - \tilde{A} \right\|_{W^{\ell,p}(M_k)} \leq C_\ell.$$

Now consider the subsequence $\nu_{k,i} := N + i$ and the gauge transformations $u^{k,i} := u^{N+i}|_{M_k} \in \mathcal{G}^{2,p}(M_k)$: These satisfy the required uniform bound for proposition 7.6 since for all $\ell \in \mathbb{N}$

$$\begin{aligned}
&\sup_{i\in\mathbb{N}} \left\| u^{k,i\,*}\tilde{A}^{\nu_{k,i}} - \tilde{A} \right\|_{W^{\ell,p}(M_k)} \\
&\leq \sup_{\nu\geq N} \left\| u^{\nu\,*}\tilde{A}^\nu - \tilde{A} \right\|_{W^{\ell,p}(M_k)} \\
&\leq \sup\Big\{ \left\| u^{N\,*}\tilde{A}^N - \tilde{A} \right\|_{W^{\ell,p}(M_k)}, \ldots, \left\| u^{N_\ell - 1\,*}\tilde{A}^{N_\ell - 1} - \tilde{A} \right\|_{W^{\ell,p}(M_k)}, C_\ell \Big\} \\
&< \infty.
\end{aligned}$$

So proposition 7.6 with $I = \mathbb{N}$ provides a subsequence $(\nu_i)_{i\in\mathbb{N}}$ and a sequence of gauge transformations $u^i \in \mathcal{G}^{2,p}_{loc}(P)$ such that

$$\sup_{i\in\mathbb{N}} \left\| u^{i\,*}\tilde{A}^{\nu_i} - \tilde{A} \right\|_{W^{\ell,p}(M_k)} < \infty \qquad \forall k, \ell \in \mathbb{N}.$$

Finally, a diagonal sequence of the $u^{i\,*}\tilde{A}^{\nu_i}$ converges uniformly with all derivatives on all compact subsets of M: For every $k \in \mathbb{N}$ the Sobolev embedding $W^{k+2,p} \hookrightarrow \mathcal{C}^k$ is compact over M_k. Hence a subsequence of these connections converges in the uniform $\mathcal{C}^k$-topology on M_k. So by fixing one further element of the sequence in every step for $k \in \mathbb{N}$ we obtain a subsequence of the $u^{i\,*}\tilde{A}^{\nu_i}$ that converges in the uniform $\mathcal{C}^k$-topology on M_k for all $k \in \mathbb{N}$. Since every compact subset of M lies in some M_k this proves the claimed convergence on all compact subsets. □

Part IV

Appendix

Appendix A

Gauge Theory

In order to set up notation and state some general facts that are used in this book we give a short introduction to connections and curvature on principal G-bundles.

Throughout, G will be a Lie group. So we first introduce the notation for the natural representations of G and its Lie algebra $\mathfrak{g} = \mathrm{T}_{\mathbb{1}}G$. Firstly, G is represented on itself by the conjugation $c : G \to \mathrm{Aut}(G)$ given by

$$c_g(h) = ghg^{-1} \qquad \forall\, g, h \in G.$$

The adjoint representation on the Lie algebra, $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g = \mathrm{d}_{\mathbb{1}} c_g$ is given by

$$\mathrm{Ad}_g(\xi) = g\xi g^{-1} \qquad \forall \xi \in \mathfrak{g}, g \in G.$$

This uses the following notation: For $\xi \in \mathfrak{g}$ and $g \in G$

$$g\xi \;:=\; \mathrm{d}_{\mathbb{1}} L_g(\xi) \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, g \exp(t\xi) \;\in \mathrm{T}_g G.$$

Here L_g denotes left multiplication by g and $\exp : \mathfrak{g} \to G$ is the usual exponential map (with respect to any metric on G). The notation ξg is defined analogously by right multiplication. This makes particular sense when $G \subset \mathbb{C}^{n\times n}$ is a matrix group since then $g\xi$ can be understood as matrix multiplication.

Next, the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is given by the Lie bracket of vector fields: We identify the Lie algebra elements $\xi \in \mathfrak{g}$ with left invariant vector fields $g \mapsto g\xi$ on G, then for $\xi, \zeta \in \mathfrak{g}$

$$\mathrm{ad}_\xi(\zeta) \;:=\; \mathrm{d}_{\mathbb{1}}\mathrm{Ad}(\xi)\,\zeta \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(t\xi)\zeta\exp(t\xi)^{-1} \;=\; \mathcal{L}_\xi\zeta(\mathbb{1}) \;=\; [\xi, \zeta].$$

In the case of a matrix group note that the Lie bracket is given by the commutator $[\xi, \zeta] = \xi\zeta - \zeta\xi$.

The underlying object in gauge theory is a **principal G-bundle** $\pi : P \to M$. This is a manifold P with a free right action $P \times G \to P$, $(p, g) \mapsto pg$ of a Lie group G such that the orbits of this action are the fibres $\pi^{-1}(x) \cong G$ of a locally trivial fibre bundle $\pi : P \to M$. Here M is a smooth manifold and the G-action preserves the fibres, i.e. $\pi(pg) = \pi(p)$.

If we denote the action of G on P by $\Theta : G \to \mathrm{Diff}(P)$, $g \mapsto \Theta_g$, then G acts on $\mathrm{T}P$ via $\mathrm{d}\Theta_g$. We simply write for all $v \in \mathrm{T}_pP$ and $g \in G$

$$vg := \mathrm{d}_p\Theta_g(v).$$

The linearization of Θ along G gives rise to the **infinitesimal action** of the Lie algebra: Every $\xi \in \mathfrak{g}$ defines a section of $\mathrm{T}P$ as follows. For all $p \in P$

$$p\xi \;:=\; \mathrm{d}_{\mathbb{1}}\Theta(\xi)\, p \;=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} p\exp(t\xi).$$

The local triviality of $\pi : P \to M$ means that there is a bundle atlas $M = \bigcup_{\alpha\in A} U_\alpha$ with equivariant local trivializations

$$\Phi_\alpha : \begin{array}{ccc} \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times G \\ p & \longmapsto & (\pi(p), \phi_\alpha(p)) \end{array} .$$

More precisely, the Φ_α are diffeomorphisms and their second component is equivariant, $\phi_\alpha(pg) = \phi_\alpha(p)g$.

This atlas gives rise to **transition functions** $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ defined by $\Phi_\alpha \circ \Phi_\beta^{-1}(x, g) = (x, \phi_{\alpha\beta}(x)g)$ for $x \in U_\alpha \cap U_\beta$, i.e. $\phi_{\alpha\beta}(x) = \phi_\alpha(p)\phi_\beta(p)^{-1}$ for all $p \in \pi^{-1}(x)$. These satisfy the cocycle conditions

$$\phi_{\alpha\alpha} \equiv \mathbb{1} \quad \text{and} \qquad \phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma} \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma. \tag{A.1}$$

In turn, given a covering $M = \bigcup_{\alpha\in A} U_\alpha$, every set of functions $(\phi_{\alpha\beta})_{\alpha,\beta\in A}$ that satisfies these conditions uniquely defines a principal G-bundle

$$\pi : \big\{[\alpha, x, g] \;\big|\; \alpha \in A, x \in U_\alpha, g \in G\big\} \to M.$$

Here $[\alpha, x, g]$ is the equivalence class of (α, x, g) with respect to the equivalence relation $(\alpha, x, g) \sim (\beta, x, \phi_{\beta\alpha}(x)g)$ and π is defined by $\pi([\alpha, x, g]) = x$. The action of G is then given by right multiplication in the last component.

If the $\phi_{\alpha\beta}$ are the transition functions of a G-bundle then this reconstructs the bundle. If two different sets of transition functions are given, then the question arises whether they might define isomorphic bundles (in the sense of G-bundle isomorphisms defined below). The answer is the following (without proof).

Lemma A.1 *Let $M = \bigcup_{\alpha\in A} U_\alpha$ be an open covering of a manifold M and let $(\phi_{\alpha\beta})_{\alpha,\beta\in A}$, $(\psi_{\alpha\beta})_{\alpha,\beta\in A}$ be two sets of Lie group valued functions on the intersections $U_\alpha \cap U_\beta$ that satisfy the cocycle conditions (A.1). Then these define isomorphic principal G-bundles over M if and only if there exists a cover $M = \bigcup_{\alpha\in A} V_\alpha$ with $V_\alpha \subset U_\alpha$ and functions $\rho_\alpha : V_\alpha \to G$ such that $\psi_{\alpha\beta}(x) = \rho_\alpha(x)\phi_{\alpha\beta}(x)\rho_\beta^{-1}(x)$ for all $x \in V_\alpha \cap V_\beta$.*

If the $\phi_{\alpha\beta}$ in this lemma arise from a bundle atlas $(U_\alpha, \Phi_\alpha)_{\alpha\in A}$ of a principal G-bundle and the $\psi_{\alpha\beta}$ meet the condition for defining the same bundle then we can think of them as arising from an atlas with different trivializations, namely $\Psi_\alpha : p \mapsto (\pi(p), \rho_\alpha(\pi(p))\phi_\alpha(p))$ on $\pi^{-1}(V_\alpha)$.

Now for any other manifold F with a representation $\rho : G \to \mathrm{Diff}(F)$ the **associated bundle $\mathbf{P} \times_\rho \mathbf{F}$** is the set of equivalence classes $\lfloor p, f\rfloor$ in $P \times F$, where the equivalence is given by ρ, i.e. $\lfloor p, f\rfloor \sim \lfloor pg, \rho(g^{-1})f\rfloor$ for all $g \in G$. (Here we write $\lfloor\cdot,\cdot\rfloor$ for the equivalence classes in order to distinguish this notation from the Lie bracket $[\cdot,\cdot]$.) With the projection $\tilde{\pi}\lfloor p, g\rfloor = \pi(p)$ this is a principal bundle over M with fibre F. A local trivialization $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ of P naturally induces the following local trivialization of $P \times_\rho F$:

$$\tilde{\Phi}_\alpha : \begin{array}{ccc} \tilde{\pi}^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ \lfloor p, f\rfloor & \longmapsto & (\pi(p), \rho(\phi_\alpha(p))f) \end{array} .$$

We will, for example, encounter the associated bundle $P \times_c G$ where F is the group G itself and the representation is the conjugation c. Another important example is $\mathfrak{g}_\mathbf{P} := \mathbf{P} \times_{\mathrm{Ad}} \mathfrak{g}$.

Next, a **G-bundle isomorphism** is a bundle isomorphism that preserves the action of the Lie group G, and such isomorphic bundles are usually identified. So when studying a fixed bundle we also have to consider its G-bundle automorphisms, i.e. diffeomorphisms $\psi : P \to P$ such that $\pi\circ\psi = \psi$ and that are equivariant, $\psi(pg) = \psi(p)g$ for all $p \in P, g \in G$.

Every G-bundle isomorphism is given by a unique element of the gauge group. The **gauge group** $\mathcal{G}(P)$ consists of the smooth maps $u : P \to G$ that are equivariant, i.e.

$$u(pg) = g^{-1}u(p)g \qquad \forall\, p \in P,\ g \in G.$$

The G-bundle isomorphism associated to an element $u \in \mathcal{G}(P)$ of the gauge group is given by $\psi(p) = p\,u(p)$. So obviously, composition $\psi_2\circ\psi_1$ of G-bundle isomorphisms corresponds to group multiplication u_1u_2 of the corresponding gauge transformations. Moreover, the gauge group is isomorphic to the group of sections of the associated bundle $P \times_c G$. Let $u \in \mathcal{G}(P)$, then the corresponding section $\bar{u} : M \to P \times_c G$ is given by

$$\bar{u}(\pi(p)) = \lfloor p, u(p)\rfloor \qquad \forall\, p \in P.$$

In the local trivialization a gauge transformation $u \in \mathcal{G}(P)$ is represented by $u_\alpha = \tilde{\phi}_\alpha\circ\bar{u} : \tilde{\pi}^{-1}(U_\alpha) \to G$ and acts by $(x, g) \mapsto (x, g\,u_\alpha(x))$ on $U_\alpha \times G$. Here $\tilde{\phi}_\alpha(\lfloor p, g\rfloor) = \phi_\alpha(p)\,g\,\phi_\alpha(p)^{-1}$ is the second component of the trivialization $\tilde{\Phi}_\alpha$ of $P \times_c G$. Thus $u_\alpha(x) = \phi_\alpha(p)u(p)\phi_\alpha(p)^{-1}$ for all $p \in \pi^{-1}(x)$, and hence on $U_\alpha \cap U_\beta$ one has the transition identity

$$u_\beta = \phi_{\alpha\beta}^{-1}u_\alpha\phi_{\alpha\beta}.$$

In turn, every such collection of G-valued functions $(u_\alpha)_{\alpha\in A}$ uniquely defines a gauge transformation $u : p \mapsto \phi_\alpha(p)^{-1}u_\alpha(\pi(p))\phi_\alpha(p)$.

Finally, to introduce connections we first note that the G-bundle P has a canonical vertical subbundle $V \subset \mathrm{T}P$: The vertical space $V_p = \ker(\mathrm{d}_p\pi) \subset \mathrm{T}_pP$ at $p \in P$ is composed of all tangencies (i.e. $p\xi$ with $\xi \in \mathfrak{g}$) to the orbits of G through p. Every complement of V_p is isomorphic to $\mathrm{im}(\mathrm{d}_p\pi) = \mathrm{T}_{\pi(p)}M$, but there is no canonical choice of this horizontal space in T_pP. Now a connection of P defines such an equivariant horizontal distribution $H \subset \mathrm{T}P$:

A **connection** on P is an equivariant $\mathfrak{g}$-valued 1-form with fixed values in the vertical direction, i.e. $A \in \Omega^1(P;\mathfrak{g})$ satisfies

$$\begin{aligned} A_{pg}(vg) &= g^{-1}A_p(v)g \qquad && \forall\, v \in \mathrm{T}_pP,\ g \in G, \\ A_p(p\xi) &= \xi && \forall\, p \in P,\ \xi \in \mathfrak{g}. \end{aligned}$$

We denote the set of smooth connections by $\mathcal{A}(P)$. Every connection $A \in \mathcal{A}(P)$ corresponds to a splitting $\mathrm{T}P = V \oplus H$, where the horizontal distribution H is defined by $H_p = \ker A_p$.

Again, this can be formulated equivalently in terms of an associated bundle: If we fix one connection $\tilde{A} \in \mathcal{A}(P)$ then the space of connections is the affine space $\mathcal{A}(P) = \tilde{A} + \Omega^1_{\mathrm{Ad}}(P;\mathfrak{g})$. Here $\Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ denotes the space of equivariant horizontal k-forms, i.e. $\tau \in \Omega^k(P;\mathfrak{g})$ that satisfy

$$\begin{aligned} \Theta_g^*\tau &= g^{-1}\tau g \qquad && \forall\, g \in G, \\ \iota_{p\xi}\tau_p &= 0 && \forall\, p \in P,\ \xi \in \mathfrak{g}. \end{aligned}$$

Now this space is isomorphic to the space $\Omega^k(M;\mathfrak{g}_P)$ of k-forms on M with values in the associated bundle $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$: For $\tau \in \Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ the corresponding $\bar{\tau} \in \Omega^k(M;\mathfrak{g}_P)$ is uniquely defined by

$$\lfloor p, \tau_p(Y_1, \dots Y_k)\rfloor = \bar{\tau}_{\pi(p)}(\mathrm{d}_p\pi(Y_1), \dots \mathrm{d}_p\pi(Y_k)) \qquad \forall\, Y_1, \dots Y_k \in \mathrm{T}_pP.$$

Consequently, every $\tau \in \Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ is represented by $\tau_\alpha = \tilde{\phi}_\alpha \circ \bar{\tau} \in \Omega^k(U_\alpha;\mathfrak{g})$ in the local trivialization. Here $\tilde{\phi}_\alpha(\lfloor p, \xi\rfloor) = \phi_\alpha(p)\,\xi\,\phi_\alpha(p)^{-1}$ is the second component of the associated trivialization of $\mathfrak{g}_P$. On the intersection $U_\alpha \cap U_\beta$ of two charts these k-forms satisfy

$$\tau_\beta = \phi_{\alpha\beta}^{-1}\,\tau_\alpha\,\phi_{\alpha\beta}. \tag{A.2}$$

The global k-form can then be reconstructed from any such set $(\tau_\alpha)_{\alpha\in A}$ by

$$\tau(Y_1, \dots Y_k) = \phi_\alpha(p)^{-1}\tau_\alpha(\mathrm{d}_p\pi(Y_1), \dots \mathrm{d}_p\pi(Y_k))\,\phi_\alpha(p) \qquad \forall\, Y_1, \dots Y_k \in \mathrm{T}_pP.$$

In the case of connections the analogous local representation depends on the chosen connection $\tilde{A}$ and there is no canonical choice for this reference connection. However, locally on $\pi^{-1}(U_\alpha)$ a natural choice of the reference connection is $\tilde{A}_\alpha = \phi_\alpha^{-1}\mathrm{d}\phi_\alpha$. This corresponds to the pullback of the splitting under $\Phi_\alpha : P \to U_\alpha \times G$. The local representative $A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ of $A \in \mathcal{A}(P)$ is then given by

$$A_\alpha(\mathrm{d}_p\pi(Y)) = \phi_\alpha(p)\,A(Y)\,\phi_\alpha(p)^{-1} - \mathrm{d}_p\phi_\alpha(Y)\,\phi_\alpha(p)^{-1} \qquad \forall\, Y \in \mathrm{T}_pP.$$

This is the local representation of connections that is used throughout the book. Note that the transition between different coordinate charts is different from (A.2) since the reference connection is not globally defined. So in order that a collection $(A_\alpha) \subset \Omega^1(U_\alpha; \mathfrak{g})$ is the local representation of a global connection it has to meet the following on $U_\alpha \cap U_\beta$:

$$A_\beta = \phi_{\alpha\beta}^{-1} A_\alpha \phi_{\alpha\beta} + \phi_{\alpha\beta}^{-1} \mathrm{d}\phi_{\alpha\beta}.$$

One can think of this as the effect of a local gauge transformation. So next, we define the action of the gauge group. For this purpose consider a gauge transformation $u \in \mathcal{G}(P)$ and the corresponding G-bundle automorphism $\psi : p \mapsto p\,u(p)$. The action of u on $A \in \mathcal{A}(P)$ is then defined by $u^*A := \psi^*A$. Note that for all $Y \in \mathrm{T}_pP$

$$\begin{aligned}
\psi^*A(Y) &= A_{\psi(p)}(\mathrm{d}_p\psi(Y)) \\
&= A_{pu(p)}\left(Yu(p)\right) + A_{pu(p)}\left(pu(p)\,u(p)^{-1}\mathrm{d}_p u(Y)\right) \\
&= u(p)^{-1}A_p(Y)u(p) + u(p)^{-1}\mathrm{d}_p u(Y),
\end{aligned}$$

and hence

$$u^*A = u^{-1}Au + u^{-1}\mathrm{d}u.$$

This is the connection on $\psi^*P \cong P$ that corresponds to the connection A on P. Hence u^*A and A are viewed as equivalent connections – they are **gauge equivalent**. Finally, the local formula for the gauge action is

$$(u^*A)_\alpha = u_\alpha^{-1} A_\alpha u_\alpha + u_\alpha^{-1}\mathrm{d}u_\alpha.$$

This shows that locally a gauge transformation can also be thought of as a change of the trivialization.

Connections moreover induce covariant derivatives on the associated vector bundles. In particular, a connection $A \in \mathcal{A}(P)$ defines the following covariant derivative on $\mathfrak{g}_P$:

$$\nabla_A : \begin{array}{ccc} \Gamma(\mathfrak{g}_P) & \longrightarrow & \Gamma(\mathrm{T}^*M \otimes \mathfrak{g}_P) \\ s & \longmapsto & \mathrm{d}s + [A, s]. \end{array}$$

Here and throughout, $\Gamma(\cdot)$ denotes the space of smooth sections of a bundle. For $X \in \mathrm{T}_xM$ with $Y \in \mathrm{T}_pP$ such that $\mathrm{d}_p\pi(Y) = X$ this evaluates as

$$\nabla_A s(X) \;=\; \lfloor\, p\,,\, \mathrm{d}_p s(Y) + [A(Y), s(p)]\,\rfloor \;\in (\mathfrak{g}_P)_x.$$

Here $s \in \Gamma(\mathfrak{g}_P)$ on the right hand side is to be understood as map from P to $\mathfrak{g}$. We can then use the standard construction to extend this covariant derivative to $\nabla_A : \Gamma(\otimes^k \mathrm{T}^*M \otimes \mathfrak{g}_P) \to \Gamma(\otimes^{k+1}\mathrm{T}^*M \otimes \mathfrak{g}_P)$ for all $k \in \mathbb{N}$. Let ∇ be the Levi-Civita connection on M, then for $\alpha \in \Gamma(\otimes^k \mathrm{T}^*M \otimes \mathfrak{g}_P)$ and $X_0, \dots, X_k \in \Gamma(\mathrm{T}M)$

$$\begin{aligned}
\nabla_A\alpha(X_0, \dots, X_k) = \nabla_A\left(\alpha(X_1, \dots, X_k)\right)(X_0) - \alpha(\nabla_{X_0}X_1, X_2, \dots, X_k) \\
- \dots - \alpha(X_1, \dots, X_{k-1}, \nabla_{X_0}X_k).
\end{aligned} \tag{A.3}$$

But the covariant derivative on $\mathfrak{g}_P$ can also be understood as the special case $k=0$ of the exterior derivative

$$\mathrm{d}_A : \begin{array}{ccc} \Omega^k_{\mathrm{Ad}}(P;\mathfrak{g}) & \longrightarrow & \Omega^{k+1}_{\mathrm{Ad}}(P;\mathfrak{g}) \\ \tau & \longmapsto & \mathrm{d}\tau + [A\wedge\tau]. \end{array}$$

Here $[A\wedge\tau]$ denotes the wedge product of the two forms with the Lie bracket used to combine the values in $\mathfrak{g}$. For example, for $A,B\in\Omega^1_{\mathrm{Ad}}(P;\mathfrak{g})$ and $X,Y\in \mathrm{T}_pP$

$$[A\wedge B](X,Y) = [A(X),B(Y)] - [A(Y),B(X)].$$

Now ${\mathrm{d}_A}^2$ does not vanish in general, but we obtain $\mathrm{d}_A\mathrm{d}_A\tau = [F_A\wedge\tau]$ for all $\tau\in\Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$, with the **curvature**

$$F_A \;=\; \mathrm{d}A + \tfrac{1}{2}[A\wedge A] \;\in \Omega^2_{\mathrm{Ad}}(P;\mathfrak{g}).$$

The curvature satisfies the Bianchi identity (see e.g. [J, Theorem 3.1.1])

$$\mathrm{d}_A F_A = 0.$$

Locally the exterior derivative d_A on $\tau\in\Omega^k_{\mathrm{Ad}}(P;\mathfrak{g})$ is represented by

$$(\mathrm{d}_A\tau)_\alpha = \mathrm{d}\tau_\alpha + [A_\alpha\wedge\tau_\alpha].$$

Thus for the curvature in terms of the local representatives A_α of the connection we obtain the same formula as globally,

$$(F_A)_\alpha = \mathrm{d}A_\alpha + \tfrac{1}{2}[A_\alpha\wedge A_\alpha].$$

In coordinates $(x^1,\dots,x^k)$ of U_α and dropping the subscript α

$$F_{ij} = \tfrac{\partial A_i}{\partial x^j} - \tfrac{\partial A_j}{\partial x^i} + [A_i,A_j].$$

A change of the trivialization has the effect that on $U_\alpha\cap U_\beta$ the local representatives of the curvature satisfy $(F_A)_\beta = \phi^{-1}_{\alpha\beta}(F_A)_\alpha\phi_{\alpha\beta}$. Analogously, gauge transformations act on the curvature by the adjoint action:

$$F_{u^*A} = u^{-1}F_A u \qquad \forall\, A\in\mathcal{A}(P), u\in\mathcal{G}(P).$$

This leads to a gauge invariant quantity if $\mathfrak{g}$ is equipped with an inner product that is invariant under the adjoint action of G. So from now on we restrict ourselves to compact Lie groups G because of the following theorem. Its proof can be found in [K, Prop.4.24].

Theorem A.2 *Let G be a compact Lie group and let $\mathfrak{g}$ be its Lie algebra. Then there exists an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ that is invariant under the adjoint action of the Lie group,*

$$\big\langle\, g\xi g^{-1}, g\zeta g^{-1}\,\big\rangle = \langle\,\xi,\zeta\,\rangle \qquad \forall \xi,\zeta\in\mathfrak{g}, g\in G. \tag{A.4}$$

Remark A.3

(i) The G-invariant inner product on $\mathfrak{g}$ moreover satisfies for all $\xi, \zeta, \eta \in \mathfrak{g}$

$$\langle [\xi, \eta], \zeta \rangle = \langle \xi, [\eta, \zeta] \rangle . \tag{A.5}$$

(ii) The inner product on $\mathfrak{g}$ can be rescaled in such a way that the associated norm $|\xi| = \sqrt{\langle \xi, \xi \rangle}$ satisfies for all $\xi, \zeta \in \mathfrak{g}$

$$|[\xi, \zeta]| \leq |\xi| \cdot |\zeta|. \tag{A.6}$$

(iii) The G-invariant inner product on $\mathfrak{g}$ induces a metric $\langle \cdot, \cdot \rangle_G$ on G by

$$\langle X , Y \rangle_G := \langle g^{-1}X , g^{-1}Y \rangle \qquad \forall X, Y \in \mathrm{T}_g G.$$

In this metric the left and right multiplications are isometries of G. Denote by $\exp_g$ the exponential map with base point $g \in G$ and set $\exp := \exp_{\mathbb{1}}$, then for all $\xi \in \mathfrak{g}$ and $g \in G$

$$\exp_g(g\xi) = g \exp(\xi), \qquad \exp(g^{-1}\xi g) = g^{-1} \exp(\xi) g.$$

Moreover, the geodesics are the flow lines of the left invariant vector fields, hence they are 1-parameter subgroups: for all $s, t \in \mathbb{R}$ and $\xi \in \mathfrak{g}$

$$\exp((s+t)\xi) = \exp(s\xi) \exp(t\xi).$$

(iv) The geodesic distance between $g, h \in G$ induced by the G-invariant inner product of $\mathfrak{g}$ is defined by

$$\mathrm{d}_G(g, h) := \inf\{|X| \bigm| X \in \mathrm{T}_g G, h = \exp_g(X)\}.$$

This distance is invariant under left and right multiplication due to (iii). It may be infinite if the group has several connected components.

Here (i) follows from differentiating (A.4) with $g = \exp(t\eta)$. The rescaling in (ii) is possible since $\mathfrak{g}$ is finite dimensional, hence with an orthonormal basis $e_1, \ldots, e_N$ and finite structure constants $\Gamma_{ij}^k = \langle [e_i, e_j], e_k \rangle$ one obtains for all $\xi, \zeta \in \mathfrak{g}$ and some constant C

$$|[\xi, \zeta]|^2 \;=\; |\xi^i \zeta^j [e_i, e_j]|^2 \;\leq\; \textstyle\sum_{i,j,k=1}^N |\Gamma_{ij}^k|^2 |\xi^i|^2 |\zeta^j|^2 \;\leq\; C|\xi|^2|\zeta|^2.$$

In (iii) one has used that for all $\xi \in \mathfrak{g}$ and $g \in G$ both $t \mapsto \exp_g(tg\xi)$ and $t \mapsto g\exp(t\xi)$ are geodesics with identical initial values. The same holds for $t \mapsto \exp(tg^{-1}\xi g)$ and $t \mapsto g^{-1}\exp(t\xi)g$.

A flow line $\gamma(t)$ satisfies $\dot\gamma(t) = \gamma(t)\xi$ for some $\xi \in \mathfrak{g}$. One checks the geodesic equation $\nabla_{\dot\gamma}\dot\gamma = 0$ with $Z(t) = \gamma(t)\eta$ for all $\eta \in \mathfrak{g}$:

$$\begin{aligned}\langle \nabla_{\dot\gamma}\dot\gamma , Z \rangle_G &= \mathcal{L}_{\dot\gamma} \langle \dot\gamma , \gamma\eta \rangle_G - \tfrac{1}{2}\mathcal{L}_Z \langle \dot\gamma , \dot\gamma \rangle_G - \langle \dot\gamma , [\dot\gamma, \gamma\eta] \rangle_G \\ &= \mathcal{L}_{\dot\gamma} \langle \xi , \eta \rangle - \tfrac{1}{2}\mathcal{L}_Z \langle \xi , \xi \rangle - \langle \xi , [\xi, \eta] \rangle \;=\; 0.\end{aligned}$$

Throughout this book every compact Lie group G is equipped with the metric from theorem A.2 and remark A.3. Furthermore, we fix a metric on M. This defines a volume element dvol_M and the Hodge operator $*$ on differential forms.[1] Together with the inner product of $\mathfrak{g}$ this moreover defines an inner product on the fibres of $\otimes^k \mathrm{T}^*M \otimes \mathfrak{g}_P$ for all $k \in \mathbb{N}_0$ as follows: In the first component of the fibre $\otimes^k \mathrm{T}^*_x M \otimes (\mathfrak{g}_P)_x$ one uses (B.1). On $(\mathfrak{g}_P)_x = \{ \lfloor p, \xi \rfloor \,|\, p \in \pi^{-1}(x), \xi \in \mathfrak{g} \}$ the G-invariant inner product of $\mathfrak{g}$ induces the welldefined inner product

$$\langle \lfloor p, \xi \rfloor \,,\, \lfloor p, \zeta \rfloor \rangle_{\mathfrak{g}_P} := \langle \xi \,,\, \zeta \rangle \,.$$

For $\sigma, \tau \in \Omega^k(M; \mathfrak{g}_P)$ this pointwise inner product equals the inner product of the local representatives $\sigma_\alpha, \tau_\alpha$ in every trivialization over $U_\alpha \subset M$:

$$\begin{aligned} \langle \sigma \,,\, \tau \rangle_{\Lambda^k \mathrm{T}^*M \otimes \mathfrak{g}_P} &= * \langle \sigma \wedge * \tau \rangle_{\mathfrak{g}_P} \\ &= * \langle \sigma_\alpha \wedge * \tau_\alpha \rangle_{\mathfrak{g}} \;=\; \langle \sigma_\alpha \,,\, \tau_\alpha \rangle_{\Lambda^k \mathrm{T}^*M \otimes \mathfrak{g}} \,. \end{aligned} \tag{A.7}$$

In the second and third expression the values of the differential forms are paired by the inner product indicated by the subscript. For example, for $\mathfrak{g}$-valued 1-forms $\sigma = \sigma_1 \mathrm{d}x^1 + \sigma_2 \mathrm{d}x^2$ and $\tau = \tau_1 \mathrm{d}x^1 + \tau_2 \mathrm{d}x^2$ on $\mathbb{R}^2$ one has

$$\langle \sigma \wedge \tau \rangle_{\mathfrak{g}} = \big(\langle \sigma_1, \tau_2 \rangle_{\mathfrak{g}} - \langle \sigma_2, \tau_1 \rangle_{\mathfrak{g}} \big) \mathrm{d}x^1 \wedge \mathrm{d}x^2 .$$

In the following we drop all subscripts from the inner products. Furthermore, $*$ denotes the obvious Hodge operator on $\mathfrak{g}_P$- or $\mathfrak{g}$-valued differential forms: When written in local coordinates as a sum of products of sections (or $\mathfrak{g}$-valued functions) and differential forms $\mathrm{d}x_{i_1} \wedge \ldots \mathrm{d}x_{i_k}$ the Hodge operator only operates on the differential form.

Now the curvature F_A can be viewed as a section of $\otimes^2 \mathrm{T}^*M \otimes \mathfrak{g}_P$, so the norm induced by above inner product defines a function $|F_A| : M \to \mathbb{R}$ that can be integrated to give the **Yang-Mills functional**

$$\mathcal{YM}(A) = \int_M |F_A|^2 \; \mathrm{dvol}_M .$$

Due to the invariance of the metric (A.4) this functional is gauge invariant,

$$\mathcal{YM}(u^*A) = \mathcal{YM}(A) \qquad \forall \, u \in \mathcal{G}(P).$$

Its extrema solve the **weak Yang-Mills equation**: Let $\Omega^1(\cdot)$ denote the smooth 1-forms, then

$$\int_M \langle F_A \,,\, \mathrm{d}_A \beta \rangle = 0 \qquad \forall \, \beta \in \Omega^1(M; \mathfrak{g}_P). \tag{A.8}$$

[1] Strictly speaking, the Hodge operator is welldefined only on oriented manifolds. However, it is welldefined up to a sign which comes from the choice of an orientation in the tangent space over every point. Whenever we use the Hodge operator in an expression, this will not depend on this sign, i.e. choice of this orientation.

Indeed, the space of connections is an affine space with vector space $\Omega^1(M;\mathfrak{g}_P)$. Now one has $F_{A+t\beta} = F_A + t\,\mathrm{d}_A\beta + \frac{1}{2}t^2[\beta\wedge\beta]$ for any tangent vector $\beta \in \Omega^1(M;\mathfrak{g}_P)$ at $A \in \mathcal{A}(P)$, and an extremum of the Yang-Mills functional has to satisfy

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\mathcal{YM}(A+t\beta) = 2\int_M \langle\, F_A\,,\, \mathrm{d}_A\beta\,\rangle\,.$$

When the base manifold M is compact and has no boundary then for smooth connections (A.8) is equivalent to the usual Yang-Mills equation $\mathrm{d}_A^* F_A = 0$. If the base manifold is allowed to have boundary then (A.8) for smooth connections is equivalent to the following boundary value problem that we call the **(strong) Yang-Mills equation** ,

$$\begin{cases} \mathrm{d}_A^* F_A = 0, \\ *F_A|_{\partial M} = 0. \end{cases}$$

Indeed, by lemma 9.3 these are the Euler-Lagrange equations for the Yang-Mills functional on manifolds with boundary. Here $\mathrm{d}_A^* : \Omega^k(M;\mathfrak{g}_P) \to \Omega^{k-1}(M;\mathfrak{g}_P)$ is the formally adjoint of the exterior derivative $\mathrm{d}_A : \Omega^{k-1}(M;\mathfrak{g}_P) \to \Omega^k(M;\mathfrak{g}_P)$. It is defined in the usual sense: For $\omega \in \Omega^k(M;\mathfrak{g}_P)$ and all $\beta \in \Omega^{k-1}(M;\mathfrak{g}_P)$ compactly supported in the interior of M

$$\int_M \langle\, \mathrm{d}_A^*\omega\,,\, \beta\,\rangle = \int_M \langle\, \omega\,,\, \mathrm{d}_A\beta\,\rangle\,.$$

From this one sees that $\mathrm{d}_A^* = -(-1)^{(n-k)(k-1)} * \mathrm{d}_A *$ holds on $\Omega^k(M;\mathfrak{g}_P)$ with $n = \dim M$, and locally for all $\omega \in \Omega^k(M;\mathfrak{g}_P)$

$$(\mathrm{d}_A^*\omega)_\alpha = \mathrm{d}^*\omega_\alpha - (-1)^{(n-k)(k-1)} * [A_\alpha \wedge *\omega_\alpha]. \tag{A.9}$$

The weak and strong Yang-Mills equation are preserved under gauge transformations. For the weak equation this is obvious from the gauge invariance of the Yang-Mills functional: the extrema come in gauge orbits. For the strong equation one can check that $\mathrm{d}_{u^*A}^* F_{u^*A} = u^{-1}(\mathrm{d}_A^* F_A)u$.

The Yang-Mills functional can be generalized to a gauge invariant **$\mathbf{L}^q$-energy** for all $1 \le q < \infty$: For $A \in \mathcal{A}(P)$ this is defined as the L^q-norm of the curvature,

$$\mathcal{E}(A) \;=\; \int_M |F_A|^q \;=\; \|F_A\|_q^q. \tag{A.10}$$

So far this is all well defined since we only considered smooth connections. The energy might however be infinite if the base manifold M is not compact. In that case there also are no natural Sobolev spaces of connections. So for the rest of this appendix we assume both M and G to be compact and explain how above concepts generalize to the appropriate Sobolev spaces.

Firstly, for $1 \le q < \infty$ and $k \in \mathbb{N}$ the Sobolev space of connections defined as in appendix B,

$$\mathcal{A}^{k,q}(P) = \tilde{A} + W^{k,q}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P),$$

is independent of the smooth reference connection $\tilde{A} \in \mathcal{A}(P)$. Only the corresponding Sobolev norm $\|\cdot\|_{W^{k,q}}$ on $W^{k,q}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ depends on $\tilde{A}$ unless $k = 0$. When the reference connection $\tilde{A}$ is clear from the context, then we will also use the following notation for the Sobolev norms on the affine space $\mathcal{A}^{k,q}(P)$:

$$\|A\|_{W^{k,q}} := \|A - \tilde{A}\|_{W^{k,q}(M,\mathrm{T}^*M\otimes\mathfrak{g}_P)}.$$

Now the L^q-energy (A.10) is well defined on a suitable Sobolev space of connections depending on the dimension $n := \dim M$ of the base manifold.

Lemma A.4 *Let $1 \leq q < \infty$ with $q \geq \frac{n}{2}$. Then the L^q-energy $\mathcal{E}$ is a continuous functional on $\mathcal{A}^{1,q}(P)$, and for every smooth reference connection $\tilde{A}$ there exists a constant C such that for all $A = \tilde{A} + \alpha \in \mathcal{A}^{1,q}(P)$*

$$\mathcal{E}(A)^{\frac{1}{q}} \leq \mathcal{E}(\tilde{A})^{\frac{1}{q}} + 2\|\alpha\|_{W^{1,q}} + C\|\alpha\|_{W^{1,q}}^2.$$

Moreover, for every $k \in \mathbb{N}$ there is a constant C such that for all connections $A = \tilde{A} + \alpha \in \mathcal{A}^{k,q}(P)$

$$\|F_A\|_{W^{k-1,q}} \leq \|F_{\tilde{A}}\|_{k-1,q} + C\left(\|\alpha\|_{W^{k,q}} + \|\alpha\|_{W^{k,q}}^2\right).$$

Proof: The curvature of a connection $A = \tilde{A} + \alpha \in \mathcal{A}^{1,q}(P)$ is

$$F_A = F_{\tilde{A}} + \mathrm{d}_{\tilde{A}}\alpha + \tfrac{1}{2}[\alpha \wedge \alpha].$$

Here $|\mathrm{d}_{\tilde{A}}\alpha| \leq 2\,|\nabla_{A_0}\alpha|$ due to the fact that for all $X, Y \in \mathrm{T}_xM$

$$\mathrm{d}_{\tilde{A}}\alpha(X,Y) = \nabla_{\tilde{A}}\alpha(X,Y) - \nabla_{\tilde{A}}\alpha(Y,X).$$

Moreover, calculating with an orthonormal basis $X_1, \ldots, X_n$ of T_xM and using (A.6)

$$\left|\tfrac{1}{2}[\alpha \wedge \alpha]\right|^2 = \sum_{i,j=1}^{n} \left|[\alpha(X_i), \alpha(X_j)]\right|^2 \leq \sum_{i,j=1}^{n} |\alpha(X_i)|^2|\alpha(X_j)|^2 = |\alpha|^4.$$

Now the assumption $q \geq \frac{n}{2}$ is the exact condition for the Sobolev embedding $\mathcal{A}^{1,q}(P) \hookrightarrow \mathcal{A}^{0,2q}(P)$. If this holds then $F_A \in L^q(M, \Lambda^2\mathrm{T}^*M \otimes \mathfrak{g}_P)$ and the energy is estimated by

$$\begin{aligned}\left|\mathcal{E}(A)^{\frac{1}{q}} - \mathcal{E}(\tilde{A})^{\frac{1}{q}}\right| &= \left|\left\|F_{\tilde{A}} + \mathrm{d}_{\tilde{A}}\alpha + \tfrac{1}{2}[\alpha \wedge \alpha]\right\|_q - \|F_{\tilde{A}}\|_q\right| \\ &\leq 2\|\nabla_{\tilde{A}}\alpha\|_q + \|\alpha\|_{2q}^2 \\ &\leq 2\|\alpha\|_{W^{1,q}} + C\|\alpha\|_{W^{1,q}}^2.\end{aligned}$$

The constant C is the constant from the Sobolev estimate. This inequality shows that the L^q-norm of F_A is finite and it also proves the continuity of the energy.

For the more general estimate we apply lemma B.3 to wedge products of Lie algebra valued 1-forms (above calculations illustrate how the generalization works). For this purpose choose $1 \leq r = s < \infty$ such that lemma B.3 applies with p replaced by q, and at the same time the Sobolev embedding $W^{k,q} \hookrightarrow W^{k-1,r}$ holds, i.e. $\frac{1}{r} \geq \frac{1}{q} - \frac{1}{n}$. This is possible since $q \geq \frac{n}{2} > \frac{n}{k+2}$ implies $\frac{1}{q} - \frac{1}{n} < \frac{1}{2q} + \frac{k}{2n}$. So with some finite constants C we obtain

$$\begin{aligned}\|F_A\|_{W^{k-1,q}} &\leq \|F_{\tilde{A}}\|_{k-1,q} + \|\mathrm{d}_{\tilde{A}}\alpha\|_{W^{k-1,q}} + \|\alpha \wedge \alpha\|_{W^{k-1,q}} \\ &\leq \|F_{\tilde{A}}\|_{k-1,q} + C\left(\|\alpha\|_{W^{k,q}} + \|\alpha\|^2_{W^{k-1,r}}\right) \\ &\leq \|F_{\tilde{A}}\|_{k-1,q} + C\left(\|\alpha\|_{W^{k,q}} + \|\alpha\|^2_{W^{k,q}}\right).\end{aligned}$$

□

In a local trivialization over $U \subset M$ the connections in $\mathcal{A}^{1,q}(P)$ are represented by 1-forms in

$$\mathcal{A}^{1,q}(U) := W^{1,q}(U, \mathrm{T}^*U \otimes \mathfrak{g}),$$

and the corresponding Sobolev norm is the usual $W^{1,q}$-norm on this space.

The L^q-energy of a connection restricted to $P|_U$ is also denoted by $\mathcal{E}$ and for $A \in \mathcal{A}^{1,q}(U)$ it equals

$$\mathcal{E}(A) = \|F_A\|_q^q,$$

where $F_A = \mathrm{d}A + A \wedge A \in L^q(U, \Lambda^2\mathrm{T}^*U \otimes \mathfrak{g})$ is the local representative of the curvature. The estimate in the previous lemma then locally becomes

$$\mathcal{E}(A)^{\frac{1}{q}} \;=\; \|F_A\|_q \;\leq\; 2\|A\|_{W^{1,q}} + C\|A\|^2_{W^{1,q}}. \tag{A.11}$$

The gauge action can also be defined on suitable Sobolev spaces of connections and for the Sobolev space $\mathcal{G}^{k,p}(P)$ of gauge transformations defined as in appendix B for $kp > n$. This space consist of all gauge transformations $u = s \cdot \exp(\xi)$, where $s \in \mathcal{G}(P)$ is smooth and $\xi \in W^{k,p}(M, \mathfrak{g}_P)$ is understood as equivariant map $\xi : P \to \mathfrak{g}$. In a trivialization over $U \subset M$ gauge transformations in $\mathcal{G}^{k,p}(P|_U)$ are represented by maps $u \in \mathcal{G}^{k,p}(U)$, i.e. $u = s \cdot \exp(\xi) : U \to G$ with $s \in \mathcal{C}^\infty(M, G)$ and $\xi \in W^{k,p}(M, \mathfrak{g})$.

These sets are Banach manifolds in the topological space of continuous gauge transformations and they are actual groups with continuous group operations.

Lemma A.5 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $kp > n$, then group multiplication and inversion are continuous maps on $\mathcal{G}^{k,p}(P)$.*

Proof: This is a consequence of lemma B.8 since inversion and multiplication are smooth maps on G. In order to get into the setting of that lemma one only has to use the coordinate chart definition of $\mathcal{G}^{k,p}(P)$ in lemma B.5. □

Moreover, the gauge action of $\mathcal{G}^{k,p}(P)$ is continuous on $\mathcal{A}^{k-1,p}(P)$. (In fact, both the multiplication and the gauge action of $\mathcal{G}^{k,p}(P)$ is smooth.)

Lemma A.6 *Let $k \in \mathbb{N}$ and $1 \le p < \infty$ be such that $kp > n$. Then the gauge action is a continuous map*

$$\begin{array}{ccc} \mathcal{G}^{k,p}(P) \times \mathcal{A}^{k-1,p}(P) & \longrightarrow & \mathcal{A}^{k-1,p}(P) \\ (u,A) & \mapsto & u^*A. \end{array}$$

Moreover, for every trivialization over some $U \subset M$ there exists a constant C such that for all $u \in \mathcal{G}^{k,p}(U)$ and $A \in \mathcal{A}^{k-1,p}(U)$

$$\|u^*A\|_{W^{k-1,p}} \le \|u^{-1}\mathrm{d}u\|_{W^{k-1,p}} + C\|A\|_{W^{k-1,p}}\bigl(1 + \|u^{-1}\mathrm{d}u\|_{W^{k-2,2p}}\bigr)^{k-1}.$$

Proof: By definition of $\mathcal{G}^{k,p}(P)$ and remark B.1 it suffices to prove this in all local trivializations over $U \subset M$. So let sequences $u_i \in \mathcal{G}^{k,p}(U)$ and $A_i \in \mathcal{A}^{k-1,p}(U)$ converge to $u \in \mathcal{G}^{k,p}(U)$ and $A \in \mathcal{A}^{k-1,p}(U)$ respectively. Then by definition (see lemma B.7 (iv)) the convergence of u_i also is in $\mathcal{C}^0(U,G)$ and $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ with respect to the $W^{k-1,p}$-norm. Thus in $u_i^*A_i = (u_i)^{-1}A_iu_i + (u_i)^{-1}\mathrm{d}u_i$ one immediately obtains the $W^{k-1,p}$-convergence of the first term and the L^p-convergence of the second term. In case $k=1$ this proves the first part of the lemma. It remains to show the $W^{k-1,p}$-convergence of the second term when $k \ge 2$ and hence $p \ge \frac{n}{2}$. For the inequality one starts with

$$\|u^*A\|_{W^{k-1,p}} \le \|u^{-1}\mathrm{d}u\|_{W^{k-1,p}} + \|u^{-1}Au\|_{W^{k-1,p}}.$$

In case $k=1$ the claimed inequality then follows from the invariance of the metric on $\mathfrak{g}$ under conjugation, (A.4). For $k \ge 2$ one needs further estimates on the second term. The following lemma in the case $\ell = k-1$, $\tau = A$, and $\tau_i = A_i$ then provides these estimates and proves the lemma. □

Lemma A.7 *Consider a trivialization of P over $U \subset M$, let $0 \le \ell < k$ be integers, and let $1 \le p < \infty$ be such that $kp > n$ and $p \ge \frac{n}{2}$. Then there exists a constant C such that the following holds.*

Let $u_i \in \mathcal{G}^{k,p}(U)$ converge to $u \in \mathcal{G}^{k,p}(U)$ and suppose that τ and τ_i are $W^{\ell,p}$-differential forms on U with values in $\mathfrak{g}$ such that τ_i converges to τ in the $W^{\ell,p}$-norm. Then $(u_i)^{-1}\tau_iu_i$ converges to $u^{-1}\tau u$ in the $W^{\ell,p}$-norm and for some constant C

$$\|u^{-1}\tau u\|_{W^{\ell,p}} \le C\|\tau\|_{W^{\ell,p}}\bigl(1 + \|u^{-1}\mathrm{d}u\|_{W^{\ell-1,2p}}\bigr)^{\ell}.$$

Proof: This will be shown by induction over $\ell \in \mathbb{N}_0$. For $\ell = 0$ the convergence follows from the $\mathcal{C}^0$-convergence of the u_i, and the invariance (A.4) of the metric on $\mathfrak{g}$ under conjugation provides the estimate

$$\|u^{-1}\tau u\|_p = \|\tau\|_p.$$

For the derivatives in case $\ell > 0$ first calculate with a vector field X on U at a fixed point $p \in U$

$$\begin{aligned} \nabla_X\left(u^{-1}\tau u\right) &= \nabla_X\left(\mathrm{Ad}_{u(p)^{-1}u}\left(u(p)^{-1}\tau u(p)\right)\right) \\ &= [u^{-1}\nabla_X u, u^{-1}\tau u] + u^{-1}(\nabla_X\tau)u. \end{aligned}$$

Now assume the lemma to hold for $\ell-1 \geq 0$, and let τ and τ_i be as assumed. Then the case $\ell = 0$ provides the L^p-convergence of $u_i^{-1}\tau_i u_i$ and it remains to show that

$$\nabla(u_i^{-1}\tau_i u_i) = [u_i^{-1}\nabla u_i, u_i^{-1}\tau_i u_i] + u_i^{-1}\nabla\tau_i u_i$$

converges to $\nabla(u^{-1}\tau u)$ in the $W^{\ell-1,p}$-norm. The convergence of the second term is provided by the induction hypothesis (the lemma for $\ell-1$ with τ replaced by $\nabla\tau$). The convergence of the Lie bracket is due to lemma B.3 with $r = s = 2p$ since both factors $u_i^{-1}\nabla u_i$ and $u_i^{-1}\tau_i u_i$ converge in $W^{\ell-1,2p}$. Indeed, for the first this is due to the Sobolev embedding $W^{\ell,p} \hookrightarrow W^{\ell-1,2p}$ and $\ell \leq k-1$. The convergence of the second factor follows from the lemma for $\ell - 1$ with (k,p) replaced by $(k-1,2p)$ since by above Sobolev embedding τ_i also converges in the $W^{\ell-1,2p}$-norm and u_i converges in $\mathcal{G}^{k-1,2p}(U)$. Thus

$$\begin{aligned}
\|[u_i^{-1}\nabla u_i, u_i^{-1}\tau_i u_i] - [u^{-1}\nabla u, u^{-1}\tau u]\|_{W^{\ell-1,p}}&\\
\leq \|u_i^{-1}\mathrm{d}u_i - u^{-1}\mathrm{d}u\|_{W^{\ell-1,2p}}\|u_i^{-1}\tau_i u_i\|_{W^{\ell-1,2p}}&\\
+ \|u^{-1}\nabla u\|_{W^{\ell-1,2p}}\|u_i^{-1}\tau_i u_i - u^{-1}\tau u\|_{W^{\ell-1,2p}}&\\
\underset{i\to\infty}{\longrightarrow} 0.&
\end{aligned}$$

The induction step for the estimate works by the same arguments. Denote all constants by C then

$$\begin{aligned}
\|u^{-1}\tau u\|_{W^{\ell,p}} &\leq \|u^{-1}\tau u\|_p + \|\nabla(u^{-1}\tau u)\|_{W^{\ell-1,p}}\\
&\leq \|\tau\|_p + \|u^{-1}\nabla\tau u\|_{W^{\ell-1,p}} + \|u^{-1}\mathrm{d}u\|_{W^{\ell-1,2p}}\|u^{-1}\tau u\|_{W^{\ell-1,2p}}\\
&\leq \|\tau\|_p + C\|\nabla\tau\|_{W^{\ell-1,p}}\big(1 + \|u^{-1}\mathrm{d}u\|_{W^{\ell-2,2p}}\big)^{\ell-1}\\
&\qquad + \|u^{-1}\mathrm{d}u\|_{W^{\ell-1,2p}}C\|\tau\|_{W^{\ell-1,2p}}\big(1 + \|u^{-1}\mathrm{d}u\|_{W^{\ell-2,4p}}\big)^{\ell-1}\\
&\leq C\|\tau\|_{W^{\ell,p}}\big(1 + \|u^{-1}\mathrm{d}u\|_{W^{\ell-1,2p}}\big)^{\ell}.
\end{aligned}$$

Here the Sobolev inequalites for $W^{\ell,p} \hookrightarrow W^{\ell-1,2p}$ and $W^{\ell-1,2p} \hookrightarrow W^{\ell-2,4p}$ hold due to $p \geq \frac{n}{2}$. □

This local estimate on $u^{-1}Au = u^*A - u^{-1}\mathrm{d}u$ also leads to the following compactness result that is necessary both for the existence of an Uhlenbeck gauge and a patching lemma for exhausted noncompact manifolds.

Lemma A.8 *Let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $kp > n$ and $p > \frac{n}{2}$. Let $(A^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{A}^{k-1,p}(P)$ and $(u^\nu)_{\nu\in\mathbb{N}} \subset \mathcal{G}^{k,p}(P)$ be two sequences such that both $\|A^\nu\|_{W^{k-1,p}}$ and $\|{u^\nu}^*A^\nu\|_{W^{k-1,p}}$ are uniformly bounded. Then the following holds.*

(i) In every trivialization over some domain $U_\alpha \subset M$ there is a uniform bound on $\|(u^\nu_\alpha)^{-1}\mathrm{d}u^\nu_\alpha\|_{W^{k-1,p}(U_\alpha)}$.

(ii) There exists a subsequence of the u^ν that converges in the $\mathcal{C}^0$-topology to some $u^\infty \in \mathcal{G}^{k,p}(P)$.

(iii) *If moreover* $1 \leq s < \infty$ *satisfies* $\frac{1}{s} > \frac{1}{p} - \frac{1}{n}$ *then the subsequence in (ii) can be chosen such that* $(u_\alpha^\nu)^{-1}\mathrm{d}u_\alpha^\nu$ *converges to* $(u_\alpha^\infty)^{-1}\mathrm{d}u_\alpha^\infty$ *in the* $W^{k-2,s}$*-norm for all trivializations.*

Proof: For (i) consider $u_\alpha^\nu \in \mathcal{G}^{k,p}(U_\alpha)$ and $A_\alpha^\nu \in \mathcal{A}^{k-1,p}(U_\alpha)$ but drop the subscript α. Then we have

$$(u^\nu)^{-1}\mathrm{d}u^\nu = u^{\nu\,*}A^\nu - (u^\nu)^{-1}A^\nu u^\nu,$$

where $\|A^\nu\|_{W^{k-1,p}}$ and $\|u^{\nu\,*}A^\nu\|_{W^{k-1,p}}$ are uniformly bounded. We will now iteratively obtain uniform bounds on $\|(u^\nu)^{-1}\mathrm{d}u^\nu\|_{W^{\ell,q_\ell}}$ for $\ell = 0,\dots,k-1$ and $q_\ell = 2^{k-\ell-1}p$. For $\ell = k-1$ that proves the claim since $q_{k-1} = p$.

For the iteration note that the Sobolev inequality for $W^{k-1,p} \hookrightarrow W^{\ell,q_\ell}$ holds in all cases: For $\ell = k-1$ these are the same spaces since $q_\ell = p$; for $\ell = k-2$ we have $q_\ell = 2p$ and $W^{k-1,p} \hookrightarrow W^{k-1,2p}$ holds due to $p \geq \frac{n}{2}$; and for $\ell \leq k-3$ we even have $W^{k-1,p} \hookrightarrow \mathcal{C}^\ell$.

The start $\ell = 0$ follows from the invariance of the metric under conjugation,

$$\begin{aligned}\|(u^\nu)^{-1}\mathrm{d}u^\nu\|_{q_0} &\leq \|u^{\nu\,*}A^\nu\|_{q_0} + \|A^\nu\|_{q_0}\\ &\leq C\big(\|u^{\nu\,*}A^\nu\|_{W^{k-1,p}} + \|A^\nu\|_{W^{k-1,p}}\big).\end{aligned}$$

Here C is the constant for above Sobolev inequality, and the right hand side is uniformly bounded by assumption.

Now assume the uniform bound on $\|(u^\nu)^{-1}\mathrm{d}u^\nu\|_{W^{\ell-1,q_{\ell-1}}}$ to be established for some $\ell \geq 1$, then use lemma A.7, above Sobolev inequality, and the fact that $2q_\ell = q_{\ell-1}$ to obtain

$$\begin{aligned}&\|(u^\nu)^{-1}\mathrm{d}u^\nu\|_{W^{\ell,q_\ell}}\\ &\leq \|u^{\nu\,*}A^\nu\|_{W^{\ell,q_\ell}} + \|(u^\nu)^{-1}A^\nu u^\nu\|_{W^{\ell,q_\ell}}\\ &\leq \|u^{\nu\,*}A^\nu\|_{W^{\ell,q_\ell}} + C\|A^\nu\|_{W^{\ell,q_\ell}}\big(1 + \|(u^\nu)^{-1}\mathrm{d}u^\nu\|_{W^{\ell-1,2q_\ell}}\big)^\ell\\ &\leq C\big(\|u^{\nu\,*}A^\nu\|_{W^{k-1,p}} + \|A^\nu\|_{W^{k-1,p}}\big(1 + \|(u^\nu)^{-1}\mathrm{d}u^\nu\|_{W^{\ell-1,q_{\ell-1}}}\big)^\ell.\end{aligned}$$

Here C denotes any finite constant, and the right hand side is uniformly bounded by assumption.

For (ii) and (iii) choose a finite bundle atlas (U_α, Φ_α) of P. Then due to (i) corollary B.9 applies in every trivialization, so there exist $\mathcal{C}^0$-convergent subsequences of the local representatives u_α^ν. Their limits u_α lie in $\mathcal{G}^{k,p}(U_\alpha)$, and for the same subsequence $(u_\alpha^\nu)^{-1}\mathrm{d}u_\alpha^\nu$ converges to $(u_\alpha^\infty)^{-1}\mathrm{d}u_\alpha^\infty$ in the $W^{k-2,s}$-norm. Since the atlas is finite some common subsequence of the u^ν converges in all bundle charts and thus also converges in the $\mathcal{C}^0$-topology on P to some continuous gauge transformation u. This limit connection actually lies in $\mathcal{G}^{k,p}(P)$ since the u_α are its local representatives – see the definition of this Sobolev space of sections in a fibre bundle. □

Appendix B

Sobolev Spaces

The aim of this appendix is to give a thorough definition of Sobolev spaces and norms for sections of fibre bundles. We also state all Sobolev estimates, compactness, and embedding results that are relevant for our purposes. In particular, we give a precise definition of the Sobolev spaces of connections and gauge transformations that are used in this book.

Throughout, (M, g) will be a Riemannian n-manifold that is (unless otherwise mentioned) not necessarily orientable or compact and is allowed to have a boundary.

Smooth functions with compact support can be integrated over M as follows: Choose any locally finite atlas $M = \bigcup_{\iota\in I} U_\iota$, $\Phi_\iota : U_\iota \to \mathbb{R}^n$ and a subordinate partition of unity $\sum_{\iota\in I} \psi_\iota \equiv 1|_M$, $\operatorname{supp}\psi_\iota \subset U_\iota$. Then for every $f \in \mathcal{C}^\infty(M)$ with compact support

$$\int_M f := \sum_{\iota\in I} \int_{\Phi_\iota(U_\iota)} \psi_\iota{\circ}\Phi_\iota^{-1} \cdot f{\circ}\Phi_\iota^{-1} \sqrt{|\det(\Phi_{\iota\,*} g)|}.$$

This sum is finite since it only runs over $\iota \in I$ such that U_ι intersects the compact support of f. The metric on M also induces a pointwise inner product on covariant tensors $\alpha, \beta \in \otimes^k \mathrm{T}_x^* M$ (this is independent of the local coordinates):

$$\langle\, \alpha\, ,\, \beta\,\rangle = g^{i_1 j_1} \ldots g^{i_k j_k} \alpha_{i_1 \ldots i_k} \beta_{j_1 \ldots j_k}. \tag{B.1}$$

Now we consider a vector bundle E over M that is equipped with an inner product of the fibres and a covariant derivative $\nabla^E : \Gamma(E) \to \Gamma(\mathrm{T}^*M \otimes E)$. ($\Gamma(\cdot)$ denotes the space of sections of the bundle.) This provides an inner product on $\otimes^k \mathrm{T}^* M \otimes E$ for all $k \in \mathbb{N}$. Also the covariant derivative on E and the Levi-Civita connection ∇^M of (M, g) can be combined to

$$\nabla : \Gamma(\otimes^k \mathrm{T}^* M \otimes E) \longrightarrow \Gamma(\otimes^{k+1} \mathrm{T}^* M \otimes E).$$

This is defined as follows: For every $\alpha \in \Gamma(\otimes^k \mathrm{T}^*M \otimes E)$ the subsequent expression is $\mathcal{C}^\infty(M)$-linear in $X_0, \dots, X_k \in \Gamma(\mathrm{T}M)$ hence defines a section of $\otimes^{k+1}\mathrm{T}^*M \otimes E$:

$$\begin{aligned}\nabla\alpha(X_0, \dots, X_k) = \nabla^E_{X_0}\left(\alpha(X_1, \dots, X_k)\right) - \alpha(\nabla^M_{X_0}X_1, X_2, \dots, X_k) \\ - \dots - \alpha(X_1, \dots, X_{k-1}, \nabla^M_{X_0}X_k).\end{aligned} \tag{B.2}$$

Let $\Gamma_0(E)$ denote the space of smooth sections of a vector bundle E with compact support. For $1 \le p < \infty$ the L^p-norm is defined for all $\alpha \in \Gamma_0(E)$ by

$$\|\alpha\|_p := \left(\int_M |\alpha|^p\right)^{\frac{1}{p}}.$$

Then for $1 \le p < \infty$ and $k \in \mathbb{N}_0$ the $W^{k,p}$-Sobolev norm is defined for all $\alpha \in \Gamma_0(E)$ by

$$\|\alpha\|_{W^{k,p}} := \left(\sum_{j=0}^k \|\nabla^j\alpha\|_p^p\right)^{\frac{1}{p}} = \left(\sum_{j=0}^k \int_M |\nabla^j\alpha|^p\right)^{\frac{1}{p}}.$$

Here $\nabla^j\alpha \in \Gamma(\otimes^j \mathrm{T}^*M \otimes E)$ is the j-th covariant derivative of α and $|\cdot|$ denotes the pointwise norm that is induced by the inner product on $\otimes^k\mathrm{T}^*M \otimes E$.

Finally, the **Sobolev space $\mathbf{W^{k,p}(M, E)}$ of sections of the vector bundle $\mathbf{E \to M}$** is defined as the completion of $\Gamma_0(E)$ with respect to the $W^{k,p}$-norm. This norm is then extended to the whole Sobolev space. In the case $j = 0$ this is the standard definition of the L^p-space and norm and we write $\|\cdot\|_p$ for the $W^{0,p}$-Sobolev norm. It also is apparent from the definition that all these Sobolev spaces are Banach spaces.

The definition so far already covers the case of differential forms on a manifold. Here a k-form is considered as section of $\otimes^k\mathrm{T}^*M$. The inner product on this bundle was defined in (B.1) and the covariant derivative is induced by the Levi-Civita connection on M as in (B.2), where $\nabla^E_{X_0}$ can simply be replaced by the Lie derivative $\mathcal{L}_{X_0}$.

In order to define the Sobolev spaces of connections on a principal G-bundle $P \xrightarrow{\pi} M$ we first have to fix a smooth reference connection $\tilde{A} \in \mathcal{A}(P)$ and choose a metric on M. Then every connection differs from $\tilde{A}$ by a section of $\mathrm{T}^*M \otimes \mathfrak{g}_P$, and the fibres of this bundle are equipped with an inner product as defined in (A.7). Moreover, $\tilde{A}$ determines a covariant derivative $\nabla_{\tilde{A}}$ on $\mathrm{T}^*M \otimes \mathfrak{g}_P$ as in (A.3). With this inner product and covariant derivative (determined by choices of a metric and a reference connection) the (affine) **Sobolev space of connections** is defined as

$$\mathcal{A}^{k,p}(P) := \tilde{A} + W^{k,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P).$$

If the manifold M is compact then this space is independent of the involved choices. However, the corresponding norm on $W^{k,p}(M, \mathrm{T}^*M \otimes \mathfrak{g}_P)$ always depends on the chosen metric and for $k \ge 1$ it also depends on the covariant derivative $\nabla_{\tilde{A}}$.

However, we still denote these Sobolev norms by $\|\cdot\|_{W^{k,p}}$ – the reference connection is usually clear from the context.

In a local trivialization $\Phi = \pi\times\phi : \pi^{-1}(U) \to U\times G$ over $U \subset M$ we can use the natural reference connection $\tilde{A} = \phi^{-1}\mathrm{d}\phi$ to define $\mathcal{A}^{k,p}(P|_U)$. The corresponding Sobolev norm on sections of $\mathrm{T}^*U \otimes \mathfrak{g}_P|_U$ equals the norm on $W^{k,p}(U, \mathrm{T}^*U \otimes \mathfrak{g})$ of the local representatives of the connections. We thus locally define the Sobolev space of connections as

$$\mathcal{A}^{k,p}(U) := W^{k,p}(U, \mathrm{T}^*U \otimes \mathfrak{g}).$$

So the affine space of connections on $P|_U$ is $\mathcal{A}^{k,p}(P|_U) = \phi^{-1}\mathrm{d}\phi + \mathcal{A}^{k,p}(U)$. Moreover, in the case of a noncompact base manifolds we denote by $\mathcal{A}^{k,p}_{loc}(P)$ the space of connections that restrict to $\mathcal{A}^{k,p}(P|_K)$ for all compact subsets $K \subset M$.

This completes the definition of Sobolev spaces of sections in vector bundles. Next, we note that the Sobolev embedding theorem generalizes directly to sections of vector bundles when the base manifold is compact. This is due to the fact that in this case the relevant Sobolev spaces and norms can equivalently be defined in finitely many coordinate patches and componentwise (regarding the fibre):

Remark B.1 Let the base manifold M of a vector bundle $E \xrightarrow{\pi} M$ be compact and consider a finite atlas $M = \bigcup_{i=1}^N U_i$, $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} V_i \times \mathbb{R}^m$. Here $V_i \subset \mathbb{R}^n$ is a coordinate chart of M and $\mathbb{R}^m$ is isomorphic to the fibres of E. So a section $\alpha \in \Gamma(E)$ is locally represented by $\Phi_{i\,*}\alpha : V_i \to \mathbb{R}^m$, which has m components $(\Phi_{i\,*}\alpha)_j : V_i \to \mathbb{R}$.

Then for every $1 \le p < \infty$ and $k \in \mathbb{N}_0$ the $W^{k,p}$-norm for sections of E is equivalent to the norm given by

$$\sum_{i=1}^N \sum_{j=1}^m \|(\Phi_{i\,*}\alpha)_j\|_{W^{k,p}(V_i)}$$

for all $\alpha \in \Gamma(E)$. Hence a section lies in $W^{k,p}(M, E)$ if and only if on all coordinate patches all of its components are $W^{k,p}$-functions.

In order to state the generalized Sobolev estimates we define the L^∞-norm for continuous sections $\alpha \in \mathcal{C}^0(M, \otimes^k \mathrm{T}^*M \otimes E)$ by

$$\|\alpha\|_\infty := \sup_{x\in M} |\alpha(x)|.$$

Then for all $j \in \mathbb{N}_0$ the uniform $\mathcal{C}^j$-topology for sections is determined by the $W^{j,\infty}$-norm defined as follows: For all $\alpha \in \mathcal{C}^j(M, E)$

$$\|\alpha\|_{W^{j,\infty}} := \sup_{k\le j} \|\nabla^k \alpha\|_\infty.$$

Theorem B.2 (Sobolev embeddings and estimates)
Let E be a vector bundle over a compact Riemannian n-manifold M. Let $j < k \in \mathbb{N}$ and $1 \leq p, q < \infty$.

(i) If $k - \frac{n}{p} \geq j - \frac{n}{q}$ then the inclusion

$$W^{k,p}(M,E) \hookrightarrow W^{j,q}(M,E)$$

is continuous, i.e. for all $\alpha \in W^{k,p}(M,E)$ and some constant C

$$\|\alpha\|_{W^{j,q}} \leq C\|\alpha\|_{W^{k,p}}.$$

(ii) If $k - \frac{n}{p} > j - \frac{n}{q}$ then the inclusion is a compact map

$$W^{k,p}(M,E) \hookrightarrow W^{j,q}(M,E).$$

(iii) If $k - \frac{n}{p} > j$ then the inclusion

$$W^{k,p}(M,E) \hookrightarrow \mathcal{C}^{j}(M,E)$$

is continuous, i.e. for all $\alpha \in W^{k,p}(M,E)$ and some constant C

$$\|\alpha\|_{W^{j,\infty}} \leq C\|\alpha\|_{W^{k,p}}.$$

Moreover, this inclusion is a compact map.

For real valued functions on a bounded domain of $\mathbb{R}^n$ these results can be found in e.g. [A, Thm.5.4,Thm.6.2]. [1] To see this generalization consider the equivalent Sobolev norm from remark B.1. We can apply the standard Sobolev estimates on the charts V_i to the components $(\Phi_{i*}\alpha)_j$. Since these are finitely many charts and components this transfers the Sobolev inequality to the chartwise defined norm for sections of the vector bundle. Finally, by the equivalence of the norms the inequality also holds for the original Sobolev norm.

The compactness of the embeddings generalizes in the same way: Let a sequence of sections of the vector bundle be bounded in the considered Sobolev norm and choose a finite set of charts as in remark B.1. Then in every chart all components are bounded and thus have a convergent subsequence (with respect to the other Sobolev norm) by the compact Sobolev embedding in the standard case. We can thus find a subsequence that converges with all components on every chart and hence also with respect to the equivalent global Sobolev norm.

A consequence of the Sobolev estimates are the following product inequalities on compact manifolds which hold in particular for $k \geq 1$, $r = s = p$ and $kp > n$. Recall that n is the dimension of the considered manifold M.

[1] When the domain Ω is bounded then L^q embedds into L^r for all $r \leq q$, thus the condition $p \leq q$ is not necessary in this case.

Lemma B.3 *Assume that M is compact, let $k \in \mathbb{N}_0$, and let $1 \le p, r, s < \infty$ be such that either*

$$r, s \ge p \qquad \text{and} \qquad \tfrac{1}{r} + \tfrac{1}{s} \;<\; \tfrac{k}{n} + \tfrac{1}{p}$$

or

$$r, s > p \qquad \text{and} \qquad \tfrac{1}{r} + \tfrac{1}{s} \;\le\; \tfrac{k}{n} + \tfrac{1}{p}.$$

Then there is a constant C such that for all $f \in W^{k,r}(M)$ and $g \in W^{k,s}(M)$ the product fg lies in $W^{k,p}(M)$ and satisfies

$$\|fg\|_{W^{k,p}} \le C \|f\|_{W^{k,r}} \|g\|_{W^{k,s}}.$$

Proof: This is proven by induction on $k \in \mathbb{N}_0$. For $k = 0$ this simply is the Hölder inequality together with the fact that M has finite volume:

$$\|fg\|_p \le (\operatorname{Vol} M)^\gamma \|f\|_r \|g\|_s \qquad \text{with} \quad \gamma = \tfrac{1}{p} - \tfrac{1}{r} - \tfrac{1}{s}.$$

In fact, this inequality and thus the theorem in case $k = 0$ holds under the slightly weaker assumptions $1 \le p, r, s \le \infty$ and $\frac{1}{r} + \frac{1}{s} \le \frac{1}{p}$. Now let $k \in \mathbb{N}$, assume the lemma to hold for $k-1$, and consider $f \in W^{k,r}(M)$ and $g \in W^{k,s}(M)$ with r and s as assumed. Then the induction hypothesis (the lemma for $k-1$) provides constants C' (for r and some $\tilde{s}$) and C'' (for s and some $\tilde{r}$), and Sobolev embeddings yield a further constant C such that $fg \in W^{k,p}(M)$ follows from the following estimate:

$$\begin{aligned}
\|fg\|_{W^{k,p}} &\le \|fg\|_p + \|g\nabla f\|_{W^{k-1,p}} + \|f\nabla g\|_{W^{k-1,p}} \\
&\le \|f\|_\alpha \|g\|_\beta + C'\|g\|_{W^{k-1,\tilde{s}}} \|\nabla f\|_{W^{k-1,r}} + C''\|f\|_{W^{k-1,\tilde{r}}} \|\nabla g\|_{W^{k-1,s}} \\
&\le C\|f\|_{W^{k,r}} \|g\|_{W^{k,s}}.
\end{aligned}$$

Here $1 \le \alpha, \beta < \infty$ are chosen such that the Sobolev embeddings $W^{k,r} \hookrightarrow L^\alpha$ and $W^{k,s} \hookrightarrow L^\beta$ hold:

$$\tfrac{1}{\alpha} \ge \tfrac{1}{r} - \tfrac{k}{n}, \qquad \tfrac{1}{\beta} \ge \tfrac{1}{s} - \tfrac{k}{n}.$$

At the same time we can make sure that $\frac{1}{\alpha} + \frac{1}{\beta} \le \frac{1}{p}$, so we have the above Hölder inequality $\|fg\|_p \le \|f\|_\alpha \|g\|_\beta$. This is due to the fact that both $\frac{1}{r} - \frac{k}{n}$, $\frac{1}{s} - \frac{k}{n}$, and the sum of both terms are all less than $\frac{1}{p}$.

Next, $p < \tilde{r} < \infty$ (and analogously $\tilde{s}$) is chosen such that the lemma for $k-1$ holds with $\tilde{r}$ instead of r, and at the same time there is a Sobolev embedding $W^{k,r} \hookrightarrow W^{k-1,\tilde{r}}$, i.e.

$$\tfrac{1}{p} + \tfrac{k-1}{n} - \tfrac{1}{s} \;\ge\; \tfrac{1}{\tilde{r}} \;\ge\; \tfrac{1}{r} - \tfrac{1}{n}.$$

If $s = p$ then the first inequality has to be strict. For $k \ge 2$ one finds such $\tilde{r}$ since by assumption

$$0 \;<\; \tfrac{1}{p} + \tfrac{k-1}{n} - \tfrac{1}{s} \;\ge\; \tfrac{1}{r} - \tfrac{1}{n} \;<\; \tfrac{1}{p}.$$

Here the second inequality is strict if $s = p$. In case $k = 1$ one has to find $1 \le \tilde{r} \le \infty$ such that

$$\tfrac{1}{p} - \tfrac{1}{s} \;\ge\; \tfrac{1}{\tilde{r}} \;\ge\; \tfrac{1}{r} - \tfrac{1}{n}.$$

If $\tilde{r} = \infty$ then one has to require in addition $r > n$ for the Sobolev embedding $W^{1,r} \hookrightarrow L^\infty$. Such $\tilde{r}$ can be found since by assumption

$$0 \leq \tfrac{1}{p} - \tfrac{1}{s} \geq \tfrac{1}{r} - \tfrac{1}{n}.$$

The case $\tilde{r} = \infty$ only occurs when $s = p$, but then $r > n$ is ensured by the strict inequality $\frac{1}{r} + \frac{1}{s} < \frac{1}{n} + \frac{1}{p}$. □

These estimates generalize directly to sections of vector bundles over compact base manifolds as does the following theorem.

Theorem B.4 (Banach-Alaoglu)
Let E be a vector bundle over a compact Riemannian manifold M. Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then every bounded sequence in $W^{k,p}(M, E)$ has a weakly convergent subsequence.

This sequentially weak compactness of the unit ball is a general fact for reflexive Banach spaces (see [Y, V.2,Thm.1]). In the standard case of functions on bounded domains in $\mathbb{R}^n$ the reflexivity (and even separability) of the considered Sobolev spaces is proven in e.g. [A, Thm.3.5]. The generalization works via the characterization of the Sobolev spaces in remark B.1.

So far we have only considered Sobolev spaces of sections of vector bundles. Now we move on to maps between manifolds and finally sections of fibre bundles, in particular gauge transformations. Firstly, consider maps from the manifold M to another manifold X. For this purpose fix an atlas $(U_\alpha, \phi_\alpha)_{\alpha\in A}$ and a metric on X. Also fix an embedding $\Phi : X \to \mathbb{R}^{2\ell+1}$; this exists by the Whitney theorem (see e.g. [H, Thm.2.2.14]) with $\ell = \dim X$.

Lemma B.5 *Assume that M is compact and let $k \in \mathbb{N}$ and $1 \leq p < \infty$ be such that $kp > n$. Then for $u \in \mathcal{C}^0(M, X)$ the following is equivalent:*

(i) $\phi_\alpha \circ u \in W^{k,p}(u^{-1}(U_\alpha), \mathbb{R}^\ell)$ *for all* $\alpha \in A$*;*

(ii) $\Phi \circ u \in W^{k,p}(M, \mathbb{R}^{2\ell+1})$*;*

(iii) $u = \exp_s(V)$ *for some* $s \in \mathcal{C}^\infty(M, X)$ *and* $V \in W^{k,p}(M, s^*\mathrm{T}X)$*;*

(iv) $u^{-1}\mathrm{d}u \in W^{k-1,p}(M, \mathrm{T}^*M \otimes \mathfrak{g})$ *in the case when* $X = G$ *is a compact Lie group with Lie algebra* $\mathfrak{g}$*.*

Now the **Sobolev space $\mathbf{W^{k,p}(M, X)}$ of maps to a manifold X** is defined as the set of continuous maps that satisfy the equivalent conditions in the previous lemma.

Finally, consider a general fibre bundle $X \hookrightarrow P \xrightarrow{\pi} M$ and fix a bundle atlas $(U_\alpha, \tau_\alpha)_{\alpha\in A}$. In every local trivialization $\pi \times \tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X$ over $U_\alpha \subset M$ a section u of this bundle corresponds to a map $\tau_\alpha \circ u : U_\alpha \to X$. The **Sobolev space $\mathbf{W^{k,p}(M, P)}$ of sections of the fibre bundle P** is then defined to consist of all those sections u such that $\tau_\alpha \circ u \in W^{k,p}(U_\alpha, X)$ for all $\alpha \in A$.

Remark B.6

(i) Lemma B.5 shows that condition (i) is independent of the atlas, (ii) is independent of the embedding, and (iii) is independent of the metric on X that determines the exponential map. Hence $W^{k,p}(M,X)$ is well defined and independent of these choices.

(ii) In the case when $X = G$ is a Lie group, property (iii) can be reformulated as $u = s \cdot \exp(\xi)$ for some $s \in \mathcal{C}^\infty(M,G)$ and $\xi \in W^{k,p}(M,\mathfrak{g})$ (with $\xi = s^{-1}V$). This shows that $W^{k,p}(M,G)$ is a Banach manifold modelled on $W^{k,p}(M,\mathfrak{g})$.

The topology on $W^{k,p}(M,X)$ is given by the folloging equivalent convergence criteria. This also defines a topology on the Sobolev spaces of sections of a fibre bundle via convergence in a bundle atlas.

Lemma B.7 *Assume that M is compact and let $k \in \mathbb{N}$ and $1 \le p < \infty$ be such that $kp > n$. Then for a sequence u_i and some u in $W^{k,p}(M,X)$ the following is equivalent:*

(i) u_i converges to u in the $\mathcal{C}^0$-topology and $\phi_\alpha \circ u_i$ converges to $\phi_\alpha \circ u$ with respect to the $W^{k,p}$-norm for all $\alpha \in A$;

(ii) $\Phi \circ u_i$ converges to $\Phi \circ u$ with respect to the $W^{k,p}$-norm;

(iii) There exist $s \in \mathcal{C}^\infty(M,X)$, $V \in W^{k,p}(M,s^*\mathrm{T}X)$, and for sufficiently large $i \in \mathbb{N}$ there are $V_i \in W^{k,p}(M,s^*\mathrm{T}X)$ such that $u = \exp_s(V)$, $u_i = \exp_s(V_i)$, and the V_i converge to V in the $W^{k,p}$-norm;

(iv) u_i converges to u in the $\mathcal{C}^0$-topology and $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in $W^{k-1,p}(M,\mathrm{T}^*M \otimes \mathfrak{g})$ in the case when $X = G$ is a compact Lie group with Lie algebra $\mathfrak{g}$.

The assumption $kp > n$ is necessary for these equivalences due to the following key lemma.

Lemma B.8 *Let $B \subset \mathbb{R}^n$ be a bounded domain. Let $1 \le p < \infty$ be such that $kp > n$ and fix $k, m, N \in \mathbb{N}$. Suppose that $f \in \mathcal{C}^k(\mathbb{R}^m,\mathbb{R}^N)$, then composition with f is a continuous map*

$$\begin{array}{ccc} W^{k,p}(B,\mathbb{R}^m) & \longrightarrow & W^{k,p}(B,\mathbb{R}^N) \\ u & \mapsto & f \circ u. \end{array}$$

Proof: Let $u \in W^{k,p}(B,\mathbb{R}^m)$ be given and consider $f \circ u : B \to \mathbb{R}^N$. Then for all integers $0 \le \ell \le k$ and $j_1,\dots,j_\ell = 1,\dots,n$ we have to show that $\partial_{j_1}\dots\partial_{j_\ell}(f \circ u)$ lies in $L^p(B,\mathbb{R}^N)$ and depends $W^{k,p}$-continuously on u. Every such derivative is a finite sum of terms of the form

$$(\partial_{i_1}\dots\partial_{i_\ell} f)\circ u \cdot \partial_{I_1} u^{i_1} \dots \partial_{I_\ell} u^{i_\ell} \tag{B.3}$$

with integers $0 \leq \ell \leq k$ and $1 \leq i_1, \ldots, i_\ell \leq m$ and nonempty multiindices $I_1, \ldots, I_\ell$ of integers $1, \ldots, n$ with $|I_1| + \ldots + |I_\ell| \leq k$. Here u^i denotes the i-th component of u in $\mathbb{R}^m$, ∂_I is the usual multiple derivative and $|I|$ is the number of elements in the multiindex I.

Firstly, the Sobolev embedding $W^{k,p} \hookrightarrow \mathcal{C}^0$ holds due to $kp > n$, so u is continuous. Since $\ell \leq k$ also $\partial_{i_1} \ldots \partial_{i_\ell} f$ is continuous and thus the composition of these two maps is continuous and bounded on B. More precicely, this first factor in (B.3) depends $\mathcal{C}^0$-continuously on u. Again, the continuous Sobolev embedding $W^{k,p} \hookrightarrow \mathcal{C}^0$ applies, so this factor varies $\mathcal{C}^0$-continuously with $W^{k,p}$-small variations of u.

It remains to prove that the L^p-norm of the second factor $\partial_{I_1} u^{i_1} \ldots \partial_{I_\ell} u^{i_\ell}$ in (B.3) is finite and that this factor depends $W^{k,p}$-continuously on u. This is a consequence of the Sobolev embeddings $W^{k-m,p} \hookrightarrow L^{\frac{Mp}{m}}$ for integers $1 \leq M \leq k$ and $1 \leq m \leq M$. (To check the condition $k - m - \frac{n}{p} \geq -\frac{nm}{Mp}$ for this embedding note that $k - m \geq \frac{n}{kp}(k - \frac{mM}{k})$ since $m \leq M \leq k$.) For $M = |I_1| + \ldots + |I_\ell| \leq k$ and $m = |I_\iota|$ with $\iota = 1, \ldots, \ell$ the corresponding Sobolev estimate with some constant C yields

$$\left\|\partial_{I_\iota} u^{i_\iota}\right\|_{\frac{Mp}{|I_\iota|}} \leq C \left\|\partial_{I_\iota} u^{i_\iota}\right\|_{W^{k-|I_\iota|,p}} \leq C \|u\|_{W^{k,p}}.$$

Finally, we can apply the Hölder inequality for $\frac{1}{p} = \frac{|I_1|}{Mp} + \ldots + \frac{|I_\ell|}{Mp}$ to obtain

$$\left\|\partial_{I_1} u^{i_1} \ldots \partial_{I_\ell} u^{i_\ell}\right\|_p \leq \left\|\partial_{I_1} u^{i_1}\right\|_{\frac{Mp}{|I_1|}} \cdots \left\|\partial_{I_\ell} u^{i_\ell}\right\|_{\frac{Mp}{|I_\ell|}} \leq C \|u\|_{W^{k,p}}^{\ell}$$

for some finite constant C. So the second factor in (B.3) lies in $L^p(B, \mathbb{R}^N)$, and its continuity is seen by an analogous calculation. □

This certainly extends to the case when $\mathbb{R}^n$ is replaced by an n-manifold (see remark B.1) and $\mathbb{R}^m$ by an open subset of $\mathbb{R}^m$. Then it can be used in several versions to prove the equivalences in the definition of $W^{k,p}(M, X)$ and its topology. This lemma also shows that the definition of Sobolev spaces of sections of fibre bundles is independent of the choice of the bundle atlas – all transition maps are smooth.

Proof of the lemmata B.5 and B.7 :

We only give a detailled proof of lemma B.5; the arguments for the sequences in lemma B.7 are analogous except for little details that will be mentioned. The strategy of the proof is (i)⇒(ii)⇒(iii)⇒(i). Then in the case of a compact Lie group we show in addition (iii)⇒(iv)⇒(ii). As a general fact note that $u(M) \subset X$ is compact since X was assumed to be compact and u is continuous.

(i) ⇒ (ii) :

For every $x \in X$ a neighbourhood of $\Phi(x) \subset \Phi(X)$ is parametrized by orthogonal projection to $\Phi_* \mathrm{T}_x X \subset \mathbb{R}^{2\ell+1}$. Hence there exists a neighbourhood $V \subset X$ of x with $V \subset U_\alpha$ for some $\alpha \in A$, a smooth projection $\pi : \mathbb{R}^{2\ell+1} \to \mathbb{R}^\ell$ such that

$\psi = \pi{\circ}\Phi : V \to \mathbb{R}^\ell$ is a chart of X, and a smooth map $\tilde{\Phi} : \mathbb{R}^\ell \to \mathbb{R}^{2\ell+1}$ such that $\Phi|_V = \tilde{\Phi}{\circ}\psi$. (The latter is due to the fact that $\Phi(V)$ is a graph over $\Phi_* \mathrm{T}_x X$.)

Since $u(M) \subset X$ is compact there exist a finite covering by such neighbourhoods, $u(M) \subset \bigcup_{i=1}^N V_i$, with coordinate charts $\psi_i : V_i \to \mathbb{R}^\ell$ and smooth maps $\tilde{\Phi}_i : \mathbb{R}^\ell \to \mathbb{R}^{2\ell+1}$ such that $\Phi|_{V_i} = \tilde{\Phi}_i{\circ}\psi_i$. Moreover, every V_i lies within some U_α, thus $\psi_i{\circ}u = (\psi_i{\circ}\phi_\alpha^{-1}){\circ}(\phi_\alpha{\circ}u)$ on $u^{-1}(V_i)$, and this lies in $W^{k,p}(u^{-1}(V_i), \mathbb{R}^\ell)$. Indeed, by assumption $\phi_\alpha{\circ}u \in W^{k,p}(u^{-1}(U_\alpha), \mathbb{R}^\ell)$ and $\psi_i{\circ}\phi_\alpha^{-1}$ is a smooth map on $\mathbb{R}^\ell$, so lemma B.8 applies. Now choose a smooth partition of unity $\sum_{i=1}^N h_i\Big|_{u(M)} \equiv 1$ with $h_i \in \mathcal{C}^\infty(X, [0,1])$ and $\operatorname{supp} h_i \subset V_i$, then

$$\Phi{\circ}u \;=\; \sum_{i=1}^N (h_i{\circ}u)\cdot(\Phi{\circ}u) \;=\; \sum_{i=1}^N (h_i{\circ}\psi_i^{-1}\cdot\tilde{\Phi}_i){\circ}(\psi_i{\circ}u).$$

Here all the $h_i{\circ}\psi_i^{-1}\cdot\tilde{\Phi}_i$ are smooth maps from $\psi_i(V_i) \subset \mathbb{R}^\ell$ to $\mathbb{R}^{2\ell+1}$. So again lemma B.8 applies to every single summand and proves that this finite sum lies in $W^{k,p}(M, \mathbb{R}^{2\ell+1})$.

When considering sequences of maps one only has to replace $u(M)$ for the covering by a slightly bigger but still compact domain such that for sufficiently large i all u_i take values in this domain. This is possible due to the $\mathcal{C}^0$-convergence of the u_i.

(ii) ⇒ (iii) :

This again uses the continuous Sobolev embedding $W^{k,p} \hookrightarrow \mathcal{C}^0$: It ensures that for every $\delta > 0$ there exists $\tilde{s} \in \mathcal{C}^\infty(M, \mathbb{R}^{2\ell+1})$ such that $\|\tilde{s} - \Phi{\circ}u\|_\infty \le \delta$. Choose δ sufficiently small for $\tilde{s}$ to take values in a tubular neighbourhood of $\Phi(u(M))$. Then use the tubular neighbourhood to project $\tilde{s}$ down to $\Phi(X)$ and compose it with Φ^{-1}, which yields a map $s \in \mathcal{C}^\infty(M, X)$. Since these are all smooth maps on compact sets we can still fix $\Delta > 0$ and achieve by the choice of δ that for every $p \in M$ the geodesic distance between $u(p)$ and $s(p)$ is less than Δ. (For lemma B.7 one achieves this for every u_i by also choosing i sufficiently large such that u_i is $\mathcal{C}^0$-close to u.)

If we choose $\Delta > 0$ as the injectivity radius of the exponential map with base point in $s(M)$ then for every $p \in M$

$$V(p) \;:=\; \exp_{s(p)}^{-1}(u(p)) \;\in \mathrm{T}_{s(p)}X.$$

In order to see that this defines a $W^{k,p}$-section of $s^*\mathrm{T}X$ use remark B.1 with local trivializations given by $\mathrm{d}\Phi$ composed with projections from $\mathrm{T}\mathbb{R}^{2\ell+1} \cong \mathbb{R}^{2\ell+1}$ to $\mathbb{R}^\ell$. That way it suffices to show that the map $\mathrm{d}\Phi{\circ}V$ lies in $W^{k,p}(M, \mathbb{R}^{2\ell+1})$. In order to be able to apply lemma B.8 we rewrite

$$\mathrm{d}\Phi{\circ}V \;=\; \big(\mathrm{d}\Phi{\circ}\exp_s^{-1}{\circ}\Phi^{-1}\big){\circ}\big(\Phi{\circ}u\big) \;=\; \mathrm{Exp}_{\Phi\circ s}^{-1}{\circ}\big(\Phi{\circ}u\big).$$

Here Exp is the exponential map on $\mathbb{R}^{2\ell+1}$ for a metric constructed as follows: Push the metric on X forward to $\Phi(X)$ and choose some metric on the normal bundle $\mathrm{N}(\Phi(X))$ in $\mathrm{T}\mathbb{R}^{2\ell+1}$. Then use a tubular neighbourhood diffeomorphism to push this product metric down to a neighbourhood of $\Phi(X) \subset \mathbb{R}^{2\ell+1}$. That way the images under Φ of geodesics in X are geodesics in $\mathbb{R}^{2\ell+1}$ and $\mathrm{Exp}_{\Phi(x)}|_{\mathrm{T}(\Phi(X))} = \Phi\circ\exp_x\circ\mathrm{d}\Phi^{-1}$ for all $x \in X$. In order to be able to invert this map one of course has to choose $\Delta > 0$ to be the (possibly smaller) injectivity radius of Exp on $\Phi(s(M))$. Now lemma B.8 implies the required regularity of V since $\mathrm{Exp}_{\Phi\circ s}^{-1}$ is a smooth map from an open subset in $M\times\mathbb{R}^{2\ell+1}$ to $\mathrm{T}\mathbb{R}^{2\ell+1} \cong \mathbb{R}^{2\ell+1}$ and $\mathrm{Id}|_M \times \Phi\circ u \in W^{k,p}(M, M \times \mathbb{R}^{2\ell+1})$ by assumption.

(iii) ⇒ (i) :

Let $u = \exp_s(V)$ as in (iii), then for every $\alpha \in A$ and $p \in u^{-1}(U_\alpha)$

$$(\phi_\alpha\circ u)(p) \;=\; \big(\phi_\alpha\circ\exp_{s(p)}\circ(\mathrm{d}_{s(p)}\phi_\alpha)^{-1}\big)\circ\big(\mathrm{d}\phi_\alpha\circ V\big)(p).$$

By remark B.1 $\mathrm{d}\phi_\alpha\circ V$ is a $W^{k,p}$-section of $s^*\phi_\alpha^*\mathrm{T}\mathbb{R}^\ell \cong M \times \mathbb{R}^\ell$ over $u^{-1}(U_\alpha)$ and thus can be seen as $W^{k,p}$-map from $u^{-1}(U_\alpha)$ to $\mathbb{R}^\ell$. Then the first map in above composition is a smooth map from $u^{-1}(U_\alpha) \times \mathbb{R}^\ell$ to $\mathbb{R}^\ell$. Thus lemma B.8 implies that its composition with $\mathrm{Id}|_{u^{-1}(U_\alpha)} \times \mathrm{d}\phi_\alpha\circ V$ is a $W^{k,p}$-function, that is $\phi_\alpha\circ u \in W^{k,p}(u^{-1}(U_\alpha), \mathbb{R}^\ell)$.

(iii) ⇒ (iv) :

Let $u = s\cdot\exp(\xi)$ with $s \in \mathcal{C}^\infty(M,G)$ and $\xi \in W^{k,p}(M, \mathrm{T}^*M\otimes\mathfrak{g})$ as in (iii) for the case of $X = G$ being a compact Lie group with (finite dimensional) Lie algebra $\mathfrak{g}$. Then we have to prove that $u^{-1}\mathrm{d}u(Y) \in W^{k-1,p}(M,\mathfrak{g})$ for every smooth vector field $Y : M \to \mathrm{T}M$. So rewrite

$$\begin{aligned} u^{-1}\mathrm{d}u(Y) &= \mathrm{Ad}_{\exp(\xi)}(s^{-1}\mathrm{d}s(Y)) + \exp(-\xi)\mathrm{d}_\xi\exp(\mathrm{d}\xi(Y)) \\ &= \mathrm{Ad}_{\exp}(s^{-1}\mathrm{d}s(Y))\circ\xi + E(\xi)\circ\mathcal{L}_Y\xi, \end{aligned}$$

where the map $E : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ to the space of endomorphisms of $\mathfrak{g}$ is given by $E(\xi)\eta = \exp(-\xi)\mathrm{d}_\xi\exp(\eta)$ for all $\eta \in \mathfrak{g} \cong \mathrm{T}_\xi\mathfrak{g}$. The first term in above expression has the right form for lemma B.8, which implies that this term even lies in $W^{k,p}(M,\mathfrak{g})$. The second term lies in $W^{k-1,p}(M,\mathfrak{g})$ by similar calculations as in the lemma, which we demonstrate in the case $k = 2$:

Due to $p > \frac{n}{2}$ the Sobolev embeddings $W^{2,p} \hookrightarrow \mathcal{C}^0$ implies that ξ is continuous and hence $E(\xi)$ is bounded in the operator norm on $\mathfrak{g}$, thus for some constant C

$$\|E(\xi)\circ\mathcal{L}_Y\xi\|_p \le C\|\mathcal{L}_Y\xi\|_p.$$

This is finite since $\xi \in W^{2,p}(M,\mathfrak{g})$. For the derivative of this term let Z be another smooth vector field, then

$$\|\mathcal{L}_Z(E(\xi)\circ\mathcal{L}_Y\xi)\|_p \le \|\mathrm{d}_\xi E(\mathcal{L}_Z\xi)\circ\mathcal{L}_Y\xi\|_p + \|E(\xi)\circ\mathcal{L}_Z\mathcal{L}_Y\xi\|_p.$$

The second term here is estimated as before by the $W^{2,p}$-norm of ξ. In the first term $\mathrm{d}_\xi E$ is a bounded linear map from $\mathfrak{g}$ to $\mathrm{End}(\mathfrak{g})$, hence for some constant C

$$\|\mathrm{d}_\xi E(\mathcal{L}_Z\xi)\circ\mathcal{L}_Y\xi\|_p \leq C\|\mathcal{L}_Z\xi\|_{2p}\|\mathcal{L}_Y\xi\|_{2p}.$$

So this term is bounded by the $W^{1,2p}$-norm of ξ which is finite due to the Sobolev embedding $W^{2,p} \hookrightarrow W^{1,2p}$. This proves that $u^{-1}\mathrm{d}u$ has finite $W^{2,p}$-norm. In the general case $k > 2$ these estimates can be carried out in the same style using the Sobolev estimates established in lemma B.8.

When considering sequences the $\mathcal{C}^0$-convergence follows from the continuity of the Sobolev embedding $W^{k,p} \hookrightarrow \mathcal{C}^0$ for maps to $\mathfrak{g}$.

(iv) $\Rightarrow$ (ii) :
We give the precise proof only in case $k \leq 2$, the general case is analogous. Firstly, u is continuous, so also $\Phi\circ u$ is continuous on a compact set and thus lies in $L^p(M, \mathbb{R}^{2\ell+1})$. Secondly, for every smooth vector field Y on M

$$\mathcal{L}_Y(\Phi\circ u) \;=\; \mathrm{d}_u\Phi\circ\mathrm{d}u(Y) \;=\; E(u)\circ(u^{-1}\mathrm{d}u(Y)).$$

Let $\ell_g = \mathrm{d}_{\mathbb{1}}L_g : \mathfrak{g} \to \mathrm{T}_gG$ be the differential of the left multiplication by g. Then $E : g \mapsto \mathrm{d}_g\Phi\circ\ell_g$ smoothly maps G to the homomorphisms $\mathrm{Hom}(\mathfrak{g}, \mathbb{R}^{2\ell+1})$. Since u is continuous $E(u)$ is bounded in the operator norm and thus for some constant C

$$\|\mathcal{L}_Y(\Phi\circ u)\|_p \leq C\|u^{-1}\mathrm{d}u(Y)\|_p.$$

This is finite by assumption $u^{-1}\mathrm{d}u \in W^{1,p}(M, \mathfrak{g})$. Now consider a second smooth vector field Z on M, then use the fact that $\mathrm{d}_uE\circ\ell_u$ also is bounded in the operator norm of linear maps from $\mathfrak{g}$ to $\mathrm{Hom}(\mathfrak{g}, \mathbb{R}^{2\ell+1})$ to obtain for another constant C

$$\begin{aligned}\|\mathcal{L}_Z\mathcal{L}_Y(\Phi\circ u)\|_p &\leq \big\|\mathrm{d}_uE(\mathrm{d}u(Z))\circ(u^{-1}\mathrm{d}u(Y))\big\|_p + \big\|E(u)\circ\mathcal{L}_Z(u^{-1}\mathrm{d}u(Y))\big\|_p \\ &\leq C\big(\|u^{-1}\mathrm{d}u(Z)\|_{2p}\|u^{-1}\mathrm{d}u(Y)\|_{2p} + \|\mathcal{L}_Z(u^{-1}\mathrm{d}u(Y))\|_p\big).\end{aligned}$$

This is then bounded by the $W^{1,p}$ norm of $u^{-1}\mathrm{d}u$ due to the Sobolev embedding $W^{1,p} \hookrightarrow L^{2p}$. Again, these estimates work in the same way for general $k > 2$ analogous to lemma B.8. □

Gauge transformations fit into this framework when we consider them as sections of the associated bundle $P \times_c G$. Here $P \to M$ is a principal G-bundle with a compact Lie group G and c denotes conjugation. Then for $kp > n$ we define the **Sobolev space of gauge transformations** $\mathcal{G}^{\mathbf{k},\mathbf{p}}(\mathbf{P})$ as $W^{k,p}(M, P \times_c G)$. It is naturally isomorphic to the subspace of equivariant maps in $W^{k,p}(P, G)$ and thus it consists of all gauge transformations of the form $u = s \cdot \exp(\xi)$, where s is a smooth gauge transformation and $\xi \in W^{k,p}(M, \mathfrak{g}_P)$ is viewed as equivariant map from P to $\mathfrak{g}$. So $\mathcal{G}^{k,p}(P)$ is a Banach manifold modelled on $W^{k,p}(M, \mathfrak{g}_P)$.

Again, in the case of a noncompact base manifold $\mathcal{G}^{k,p}_{loc}(P)$ denotes the space of gauge transformations that restrict to a $\mathcal{G}^{k,p}$-gauge transformation on every compact subset $K \subset M$.

In a local trivialization over $U \subset M$ gauge transformations are represented by maps $u : M \to G$, and $\mathfrak{g}_P|_U$ is isometrically identified with $\mathfrak{g}$. Thus locally the Sobolev space of gauge transformations, $\mathcal{G}^{k,p}(P|_U)$, is identified with the set $\mathcal{G}^{\mathbf{k},\mathbf{p}}(\mathbf{U})$ of maps $u = s \cdot \exp(\xi) : U \to G$ with $s \in \mathcal{C}^\infty(U, G)$ and $\xi \in W^{k,p}(U, \mathfrak{g})$. Finally, this is equivalent to $u \in \mathcal{C}^0(U, G)$ with

$$u^{-1}\mathrm{d}u \in W^{k-1,p}(U, \mathrm{T}^*U \otimes \mathfrak{g}).$$

Of course, the definition of the Sobolev spaces of sections of fibre bundles via embeddings allows to adopt all meaningfull Sobolev embedding and compactness results. Here we only note the following extension of the Banach-Alaoglu theorem to local gauge transformations.

Corollary B.9 *Let U be a compact Riemannian n-manifold and let G be a compact Lie group. Let $k \in \mathbb{N}$ and $1 \le p < \infty$ be such that $kp > n$. Then for every sequence $(u_i)_{i\in\mathbb{N}}$ in $\mathcal{G}^{k,p}(U)$ with a uniform bound on $\|u_i^{-1}\mathrm{d}u_i\|_{W^{k-1,p}}$ there exists a subsequence that converges in the $\mathcal{C}^0$-topology to a gauge transformation $u \in \mathcal{G}^{k,p}(U)$.*

Let moreover $1 \le s < \infty$ be such that $\frac{1}{s} > \frac{1}{p} - \frac{1}{n}$, then above subsequence can be chosen such that $u_i^{-1}\mathrm{d}u_i$ converges to $u^{-1}\mathrm{d}u$ in the $W^{k-2,s}$-norm.

Proof: Choose an embedding $\Phi : G \to \mathbb{R}^m$, then by lemma B.5 (estimates in the proof (iv)⇒(ii)) the sequence $\Phi\circ u_i$ is bounded in $W^{k,p}(M, \mathbb{R}^m)$. Thus by the Banach-Alaoglu theorem B.4 a subsequence converges $W^{k,p}$-weakly to some $v \in W^{k,p}(M, \mathbb{R}^m)$. Since the embedding $W^{k,p} \hookrightarrow \mathcal{C}^0$ is compact by theorem B.2 the subsequence can also be chosen to converge in the $\mathcal{C}^0$-topology. Then v takes values in $\Phi(G)$. Thus $u = \Phi^{-1}\circ v$ is well defined, lies in $\mathcal{G}^{k,p}(M, G)$, and is the $\mathcal{C}^0$-limit of the u_i.

Moreover the embedding $W^{k-1,p} \hookrightarrow W^{k-2,s}$ is compact. Hence for a further subsequence $u_i^{-1}\mathrm{d}u_i$ converges in the $W^{k-2,s}$-norm. The limit has to be $u^{-1}\mathrm{d}u$ due to the $\mathcal{C}^0$-convergence of u_i to u and since $\mathrm{d}\Phi\circ\mathrm{d}u_i$ converges to $\mathrm{d}\Phi\circ\mathrm{d}u$ in the $W^{k-2,s}$-norm (note that $\mathrm{d}\Phi$ restricted to $\mathrm{T}(\Phi(G))$ is a bijection). □

Finally, we give a proof of the trace theorem concerning the restriction of Sobolev functions to the boundary of a compact manifold.

Theorem B.10 (Trace Theorem)
Let M be a compact Riemannian n-manifold and let $1 \le p < \infty$. In case $p < n$ assume $1 \le q \le \frac{np-p}{n-p}$, and in case $p \ge n$ assume $1 \le q < \infty$. Then the restriction to the boundary ∂M is a bounded linear map from $W^{1,p}(M)$ to $L^q(\partial M)$.

Proof: First consider the case when M is an orientable manifold. Let ν be a smooth vector field on M that extends the outward unit normal to ∂M, so we have $\mathrm{d}vol_{\partial M} = \iota_\nu \mathrm{d}vol_M|_{\partial M}$. Then for all $h \in \mathcal{C}^\infty(M)$

$$\begin{aligned}
\int_{\partial M} |h|^q \,\mathrm{d}vol_{\partial M} &= \int_M \mathrm{d}\big(|h|^q \iota_\nu \mathrm{d}vol_M\big) \\
&= \int_M |h|^q \mathcal{L}_\nu \mathrm{d}vol_M + \int_M \tfrac{\partial}{\partial\nu}|h|^q \,\mathrm{d}vol_M \\
&\le C\left(\int_M |h|^q + \int_M |h|^{q-1}|\nabla h|\right) \\
&\le C\Big(\|h\|^q_{L^q(M)} + (\mathrm{Vol}M)^{\frac{1}{s}} \|h\|^{q-1}_{L^\rho(M)} \|\nabla h\|_{L^p(M)}\Big).
\end{aligned}$$

Here the constant C is finite since ν is smooth and M is compact. We have used the Hölder inequality for three factors, and we have chosen $1 \le r, s \le \infty$ such that the Sobolev embedding $W^{1,p} \hookrightarrow L^\rho$ holds for $\rho = r(q-1)$ and

$$\frac{1}{s} + \frac{1}{r} + \frac{1}{p} = 1. \tag{B.4}$$

In case $q = 1$ the factor $|h|^{q-1} \equiv 1$ can be dropped and we set $s = p^*$, where $\frac{1}{p} + \frac{1}{p^*} = 1$. In case $p > n$ and $q > 1$ one chooses $s = p^*$ and $r = \infty$. Then the Sobolev embedding holds for $\rho = \infty$. In case $p = n$ and $q > 1$ one takes $r = \max\{\frac{1}{q-1}, p^*\} > 1$. Then (B.4) defines $s \in (p^*, \infty]$ and the Sobolev embedding holds with $\rho = r(q-1) \in [1,\infty)$. In case $p < n$ and $q > 1$ one obtains the Sobolev embedding for $\rho = r(q-1) = \frac{np}{n-p}$ by choosing $r = \frac{np}{(n-p)(q-1)}$. Here we have $r \ge p^*$ due to $q \le \frac{np-p}{n-p}$, and hence (B.4) defines $s \in (p^*, \infty]$. Moreover, the Sobolev embedding $W^{1,p} \hookrightarrow L^q$ holds in all these cases due to $q \le \frac{np}{n-p}$. So in any case we find a constant C such that for all $h \in \mathcal{C}^\infty(M)$

$$\|h\|_{L^q(\partial M)} \le C\|h\|_{W^{1,p}(M)}. \tag{B.5}$$

This shows that the restriction map $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(\partial M)$, $h \mapsto h|_{\partial M}$ extends continuously to a bounded linear map ${}^{1,p}(M) \to L^q(\partial M)$.

In the case of a nonorientable manifold one can cover a neighbourhood of the boundary by finitely many domains on which the lemma is already established (compact orientable domains with smooth boundary). On each of these domains the corresponding embedding is continuous. Now the boundaries of all these domains cover the boundary of the manifold, hence the (finitely many) embeddings patch together to give the required embedding to the whole boundary. □

Appendix C

$\mathbf{L^p}$-Multipliers, Mollifiers, and Poisson Kernels

This appendix gives a collection of results concerning several types of operators on L^p-spaces. Since it fits in nicely with all the L^p-theory in this book we explain the general concept of L^p-multipliers. These are bounded operators on L^p-spaces given by multiplication of the Fourier transform with a function. There is an easy but little known criterion for L^p-multipliers by Mihlin. It is not needed in this book but we use it to give a simple proof of the Calderon-Zygmund inequality.

The Calderon-Zygmund inequality says that $f \mapsto \partial_i\partial_j(K * f)$ is a bounded linear operator on $L^p(\mathbb{R}^n)$, where K is the Poisson kernel. We moreover prove that $f \mapsto K * f$ is a bounded linear operator from $L^p(\Omega)$ to $W^{2,p}(\Omega)$ for bounded domains $\Omega \subset \mathbb{R}^n$.

Another type of operators on $L^p(\mathbb{R}^n)$ is the convolution with mollifiers. These are functions $\rho_t(x) = t^{-n}\rho(\frac{x}{t})$ for $t > 0$ and $\rho \in \mathcal{C}^\infty(\mathbb{R}^n, [0,\infty))$ with $\int \rho = 1$. The mollifier theorem asserts that ρ_t converges to the Dirac distribution as $t \to 0$, and moreover the convolution with ρ_t converges to the identity on $L^p(\mathbb{R}^n)$. In most standard textbooks this is only proven under the assumption that ρ has compact support, see e.g. [GT, Lemma 7.1,7.2]. Here we give a proof for the general case $\rho \in L^1(\mathbb{R}^n, [0,\infty))$.

The convolution with an L^1-functions indeed is a bounded linear operator on $L^p(\mathbb{R}^n)$ by Young's inequality that is proven e.g. in [L, Thm.1.2].

Theorem C.1 (Young's inequality)
Let $1 \leq p \leq \infty$ and suppose that $g \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$. Then the convolution $g * f$ lies in $L^p(\mathbb{R}^n)$ and satisfies

$$\|g * f\|_p \leq \|g\|_1 \|f\|_p.$$

Now consider $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and its Fourier transform $\widehat{g} \in L^2(\mathbb{R}^n)$. Then the convolution operator $T_g : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $f \mapsto g * f$ is also given by multiplication of the Fourier transform with $\widehat{g}$,

$$\widehat{T_g f} = \widehat{g * f} = \widehat{g} \cdot \widehat{f}.$$

(For the extension of the Fourier transformation to an isomorphism of $L^2(\mathbb{R}^n)$ see e.g. [Fo, §12.11].) Here T_g extends to a bounded linear operator on all L^p-spaces by Young's inequality. However, there are more general bounded linear operators on $L^p(\mathbb{R}^n)$ of this form with $\widehat{g}$ replaced by another function such that the operator is not necessarily also given by convolution with a function.

More precisely, for every bounded measurable function $m : \mathbb{R}^n \to \mathbb{C}$ one can define an operator T_m on $L^2(\mathbb{R}^n)$ by

$$\widehat{T_m f} = m \cdot \widehat{f}.$$

If T_m extends to a bounded operator on $L^p(\mathbb{R}^n)$ then the function m is called an **$\mathbf{L^p}$-multiplier**. The following theorem gives a useful sufficient condition for a function to be an L^p-multiplier for all $1 < p < \infty$. In the case $n = 1$ this is essentially the Marcinkiewicz multiplier theorem which was generalized to higher dimensions by Mihlin [M]. However, the particularly easy criterion (C.1) only appers in [LSU] and all other standard literature (e.g. [St2, VI §4]) uses a criterion that is much harder to test. Here we explain how to reduce the criterion of Mihlin, Ladyzhenskaya, Solonnikov and Ural'tseva to a more general criterion in [St1].

Theorem C.2 *Let $m : \mathbb{R}^n \to \mathbb{C}$ be a measurable function that for some constant A satisfies*

$$\left| x_{i_1} \dots x_{i_s} \frac{\partial^s m}{\partial x_{i_1} \dots \partial x_{i_s}} \right| \leq A \tag{C.1}$$

for all integers $0 \leq s \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_s \leq n$. Then m is an L^p-multiplier for all $1 < p < \infty$, i.e. there exists a constant C such that whenever $f \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ then $T_m f \in L^p(\mathbb{R}^n)$ and

$$\|T_m f\|_p \leq C \|f\|_p.$$

Proof: The condition for $s = 0$ asserts that m is bounded. This is assumption (a) of [St1, IV §6,Thm.6']. It remains to check assumptions (b) and (c), that is

$$\int_\rho \left| \frac{\partial^s m}{\partial x_{i_1} \dots \partial x_{i_s}} \right| \mathrm{d}x_{i_1} \dots \mathrm{d}x_{i_s} \leq B$$

for some uniform constant B and every dyadic rectangle ρ that arise as follows. For every $1 \leq s \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_s \leq n$ there is a decomposition of $\mathbb{R}^n$ into s-dimensional dyadic rectangles in the direction of $x_{i_1}, \ldots, x_{i_s}$. These rectangles are given by numbers $\sigma_1, \ldots, \sigma_s \in \{-1, 1\}$, $l_1, \ldots, l_s \in \mathbb{Z}$ and $a_{s+1}, \ldots, a_n \in \mathbb{R}$: The corresponding dyadic rectangle is

$$\rho = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} x_{i_j} \in \left[\sigma_j 2^{l_j}, \sigma_j 2^{l_j+1}\right], & j = 1, \ldots, s, \\ x_{i_j} = a_j, & j = s+1, \ldots, n \end{array} \right\},$$

where $(i_j)_{j=1,\dots,n} = \{1,\dots,n\}$. By assumption on every such rectangle

$$\left|\frac{\partial^s m}{\partial x_{i_1}\dots\partial x_{i_s}}\right| \le \frac{A}{|x_{i_1}\dots x_{i_s}|} = \frac{A}{\sigma_1 x_{i_1}\dots\sigma_s x_{i_s}},$$

and hence independent of the signs $\sigma_1,\dots,\sigma_s$

$$\begin{aligned}\int_\rho \left|\frac{\partial^s m}{\partial x_{i_1}\dots\partial x_{i_s}}\right| \mathrm{d}x_{i_1}\dots\mathrm{d}x_{i_s} &\le \int_{2^{l_1}}^{2^{l_1+1}}\dots\int_{2^{l_s}}^{2^{l_s+1}} \frac{A}{x_{i_1}\dots x_{i_s}}\,\mathrm{d}x_{i_1}\dots\mathrm{d}x_{i_s}\\ &= A\prod_{j=1}^{s}\int_{2^{l_j}}^{2^{l_j+1}}\frac{1}{x_{i_j}}\,\mathrm{d}x_{i_j}\\ &= A(\ln 2)^s \le A(\ln 2)^n.\end{aligned}$$

□

This criterion provides a proof of the Calderon-Zygmund inequality.

Theorem C.3 (Calderon-Zygmund inequality)
For every $1 < p < \infty$ and $n \in \mathbb{N}$ there exists a constant C such that

$$\|\nabla^2 u\|_p \le C\|\Delta u\|_p$$

holds for all $u \in W^{2,p}(\mathbb{R}^n)$.

Proof: It suffices to consider $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ with compact support since this is a dense subset of $W^{2,p}(\mathbb{R}^n)$ and the inequality is preserved under $W^{2,p}$-limits. For every such function u and integers $1 \le k,l \le n$ the standard rules for Fourier transforms give

$$\widehat{\partial_k\partial_l u}(x) = -x_k x_l\,\widehat{u}(x), \qquad \widehat{-\Delta u}(x) = -|x|^2\,\widehat{u}(x).$$

The function $m(x) = \frac{x_k x_l}{|x|^2}$ is measurable as a composition of measurable functions and it is bounded: $|m(x)| \le 1$ for $x \ne 0$ since $|x_k|/|x| \le 1$ holds for all k. Thus the operator T_m is well defined and maps $f = -\Delta u$ to $T_m f = \partial_k\partial_l u$. We shall see that m is an L^p-multiplier, hence there exists a constant C such that

$$\|\partial_k\partial_l u\|_p = \|T_m f\|_p \le C\|f\|_p = C\|\Delta u\|_p.$$

Since this holds for all second derivatives of u it proves the claim. So it remains to check the condition (C.1) of theorem C.2 for m to be an L^p-multiplier.

The criterion for $s = 0$ is satisfied since m is bounded. For $s = 1$ and $i \ne k,l$ the criterion is met due to

$$\left|x_i\frac{\partial m}{\partial x_i}\right| = \left|-2\,\frac{x_k x_l x_i^2}{|x|^4}\right| \le 2$$

and similarly, for all numbers $1 \leq i_1 < i_2 < \ldots < i_s \leq n$ not containing k or l one calculates

$$\left| x_{i_1} \ldots x_{i_s} \frac{\partial^s m}{\partial x_{i_1} \ldots \partial x_{i_s}} \right| = \left| (-2)^s s! \, \frac{x_k x_l x_{i_1}^2 \ldots x_{i_s}^2}{|x|^{2(s+1)}} \right| \leq 2^s s!.$$

Then in case $k = l$ the criterion including an x_k-derivative also is fullfilled:

$$\begin{aligned} \left| x_k x_{i_1} \ldots x_{i_s} \frac{\partial^{s+1} m}{\partial x_k \partial x_{i_1} \ldots \partial x_{i_s}} \right| &= \left| (-2)^{s+1} s! \frac{((s+1)x_k^2 - |x|^2) x_k^2 x_{i_1}^2 \ldots x_{i_s}^2}{|x|^{2(s+2)}} \right| \\ &\leq 2^{s+1}(s+1)!\,. \end{aligned}$$

If $k \neq l$

$$\begin{aligned} \left| x_k x_{i_1} \ldots x_{i_s} \frac{\partial^{s+1} m}{\partial x_k \partial x_{i_1} \ldots \partial x_{i_s}} \right| &= \left| (-2)^s s! \, \frac{(|x|^2 - 2(s+1)x_k^2) x_k x_l x_{i_1}^2 \ldots x_{i_s}^2}{|x|^{2(s+2)}} \right| \\ &\leq 2^{s+1}(s+1)!, \end{aligned}$$

and the same inequality holds if an x_l-derivative is included instead of x_k. Finally, if $k \neq l$ and both derivatives are included one checks

$$\left| x_l x_k x_{i_1} \ldots x_{i_s} \frac{\partial^s m}{\partial x_l \partial x_k \partial x_{i_1} \ldots \partial x_{i_s}} \right| = \ldots \leq 2 \cdot 2^{s+2}(s+2)!\,.$$

This covers all cases of distinct derivatives and thus m is an L^p-multiplier as claimed – the criterion (C.1) is met with $A = 2^{n+3}(n+2)!$. □

The multipliers $m(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}$ considered in this proof are the Fourier transformed second derivatives of the Poisson kernel (the fundamental solution of the negative definite Laplacian $-\Delta$)

$$K(z) = \begin{cases} \frac{1}{\omega_2} \log |z| & \text{if } n = 2, \\ -\frac{1}{(n-2)\omega_n} |z|^{-(n-2)} & \text{if } n \geq 3. \end{cases}$$

Here $\omega_n = 2\pi^{\frac{n}{2}} \Gamma(\frac{n}{2})^{-1}$ is the volume of the unit sphere in $\mathbb{R}^n$. In general, for $f \in L^p(\mathbb{R}^n)$ the solution $u = K * f$ of $\Delta u = f$ does not lie in $W^{2,p}(\mathbb{R}^n)$ though its second derivatives are L^p-functions. For compactly supported functions, however, this is the case.

Theorem C.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $1 < p < \infty$. Then for every $f \in L^p(\mathbb{R}^n)$ with support in Ω the convolution $K * f$ satisfies $\Delta(K * f) = f$. Moreover, $(K * f)|_\Omega \in W^{2,p}(\Omega)$ and there exists a constant depending on Ω, n and p such that for all $f \in L^p(\mathbb{R}^n)$ supported in Ω*

$$\|K * f\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

Proof: Let $f \in L^p(\mathbb{R}^n)$ be given that vanishes outside of Ω. As established in the previous proof of the Calderon-Zygmund inequality $f \mapsto \partial_i\partial_j(K * f)$ is given by an L^p-multiplier, hence all second derivatives of $K * f$ lie in $L^p(\mathbb{R}^n)$ and satisfy

$$\|\partial_i\partial_j(K*f)\|_{L^p(\Omega)} \;\leq\; \|\partial_i\partial_j(K*f)\|_{L^p(\mathbb{R}^n)} \;\leq\; C\|f\|_{L^p(\mathbb{R}^n)} \;=\; C\|f\|_{L^p(\Omega)}.$$

Here $i, j = 1, \dots, n$ and the constant C only depends on n and p. It remains to prove a similar inequality for $K * f$ and its first derivatives. For these the constant will depend on the size of the support of f, which is why we had to fix a bounded domain Ω.

The kernel K as well as its first derivatives $\partial_i K$ is integrable on compact domains: For $n = 2$ and $n \geq 3$ respectively the integral over the ball of radius R is

$$\int_{B_R(0)} |K(z)| \mathrm{d}^2 z \;=\; \int_0^R |\log r| r \mathrm{d}r \;=\; \tfrac{1}{4} R^2 |2\log R - 1|,$$

$$\int_{B_R(0)} |K(z)| \mathrm{d}^n z \;=\; \tfrac{1}{n-2} \int_0^R r^{-(n-2)} r^{n-1} \mathrm{d}r \;=\; \tfrac{1}{2(n-2)} R^2.$$

Concerning the first derivatives note that

$$|\partial_i K(z)| \;=\; \left| \frac{z_i}{\omega_n |z|^n} \right| \;\leq\; \frac{1}{\omega_n} |z|^{-(n-1)}$$

and hence

$$\int_{B_R(0)} |\partial_i K(z)| \mathrm{d}^n z \;=\; \int_0^R r^{-(n-1)} r^{n-1} \mathrm{d}r \;=\; R.$$

Now let $\Gamma = K$ or $\Gamma = \partial_i K$ for $i = 1, \dots, n$. Then the Hölder inequality yields

$$\begin{aligned} |(\Gamma * f)(x)| &= \left| \int_\Omega \Gamma(x-y)^{1-\frac{1}{p}} \Gamma(x-y)^{\frac{1}{p}} f(y) \,\mathrm{d}^n y \right| \\ &\leq \left(\int_\Omega |\Gamma(x-y)| \,\mathrm{d}^n y \right)^{1-\frac{1}{p}} \left(\int_\Omega |\Gamma(x-y)| \cdot |f(y)|^p \,\mathrm{d}^n y \right)^{\frac{1}{p}} \end{aligned}$$

and hence

$$\begin{aligned} &\|\Gamma * f\|_{L^p(\Omega)} \\ &\leq \left(\sup_{x\in\Omega} \int_\Omega |\Gamma(x-y)| \,\mathrm{d}^n y \right)^{1-\frac{1}{p}} \left(\int_\Omega \int_\Omega |\Gamma(x-y)| \cdot |f(y)|^p \,\mathrm{d}^n y \,\mathrm{d}^n x \right)^{\frac{1}{p}} \\ &\leq \left(\sup_{x\in\Omega} \int_\Omega |\Gamma(x-y)| \,\mathrm{d}^n y \right)^{1-\frac{1}{p}} \left(\sup_{y\in\Omega} \int_\Omega |\Gamma(x-y)| \,\mathrm{d}^n x \right)^{\frac{1}{p}} \left(\int_\Omega |f(y)|^p \,\mathrm{d}^n y \right)^{\frac{1}{p}} \\ &\leq \int_{B_R(0)} |\Gamma(x)| \,\mathrm{d}^n x \;\cdot \|f\|_{L^p(\Omega)}. \end{aligned} \tag{C.2}$$

Since Ω is bounded the above integrals of $|\Gamma(x-y)|$ can all be estimated from above by the integral of $|\Gamma|$ over some large ball $B_R(0)$. This gives the finite constant C in

$$\|K * f\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \qquad \|\nabla(K * f)\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

Thus we have seen that convolution with the Poisson kernel K is a continuous map from $L^p(\Omega)$ to $W^{2,p}(\Omega)$. It remains to show that $\Delta(K * f) = f$ holds for all $f \in L^p(\mathbb{R}^n)$ with support in Ω. This holds for smooth functions f with compact support by e.g. [GT, Lemma 4.2], and since on every bounded domain $\Delta(K * f)$ depends continuously on f in the L^p-norm the equality extends to all of $L^p(\Omega)$. □

The proof of the mollifier theorem below uses similar estimates for convolutions as the previous proof.

Theorem C.5 *Let $\rho \in L^1(\mathbb{R}^n)$ satisfy $\rho(x) \geq 0$ for almost all $x \in \mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} \rho(x)\,\mathrm{d}^n x \;=\; 1.$$

For all $t > 0$ define $\rho_t \in L^1(\mathbb{R}^n)$ by $\rho_t(x) = t^{-n}\rho(\frac{x}{t})$ for almost all $x \in \mathbb{R}^n$. Then the following holds.

(i) Let $f \in \mathcal{C}^0(\mathbb{R}^n)$ be bounded. Then for all $x \in \mathbb{R}^n$

$$(\rho_t * f)(x) \;\underset{t\to 0}{\longrightarrow}\; f(x).$$

Moreover, this convergence is uniform on all compact sets.

(ii) Let $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then

$$\rho_t * f \;\overset{L^p}{\underset{t\to 0}{\longrightarrow}}\; f.$$

Proof: For (i) let $f \in \mathcal{C}^0(\mathbb{R}^n)$ with $\|f\|_\infty < \infty$ be given. To prove the convergence on all compact sets it suffices to consider balls $B_K \subset \mathbb{R}^n$ of radius K around 0. Fix $K > 0$ and to prove the convergence choose some $\varepsilon > 0$. Since f is uniformly continuous on the compact set B_K there exists $\delta > 0$ such that

$$|f(x-y) - f(x)| \leq \tfrac{1}{2}\varepsilon \qquad \forall\, x \in B_K,\; y \in B_\delta.$$

Moreover, since $\int_{\mathbb{R}^n} \rho = 1$ and ρ is positive almost everywhere one finds $R > 0$ such that

$$\int_{\mathbb{R}^n \setminus B_r} \rho(z)\,\mathrm{d}^n z \;\leq\; \tfrac{\varepsilon}{4\|f\|_\infty} \qquad \forall r \geq R.$$

So we obtain for all $t \leq \sup\{1, \frac{\delta}{R}\}$ and all $x \in B_K$

$$\int_{\mathbb{R}^n \setminus B_\delta} \rho_t(y)\,\mathrm{d}^n y \;=\; \int_{\mathbb{R}^n \setminus B_\delta} \rho(\tfrac{y}{t}) t^{-n}\,\mathrm{d}^n y \;=\; \int_{\mathbb{R}^n \setminus B_{\frac{\delta}{t}}} \rho(z)\,\mathrm{d}^n z \;\leq\; \tfrac{\varepsilon}{4\|f\|_\infty}.$$

This proves the uniform convergence: For the given $\varepsilon > 0$ we found constants $\delta > 0$ and R such that for all $x \in B_K$ and all $t \leq \sup\{1, \frac{\delta}{R}\}$

$$\begin{aligned}
&\big|(\rho_t * f)(x) - f(x)\big| \\
&= \left|\int_{\mathbb{R}^n} \rho_t(y)\big(f(x-y) - f(x)\big)\mathrm{d}^n y\right| \\
&\leq \int_{B_\delta} \rho_t(y)\big|f(x-y) - f(x)\big|\mathrm{d}^n y \;+ \int_{\mathbb{R}^n\setminus B_\delta} \rho_t(y)\big|f(x-y) - f(x)\big|\mathrm{d}^n y \\
&\leq \tfrac{1}{2}\varepsilon \int_{B_\delta} \rho_t(y)\,\mathrm{d}^n y \;+\; 2\|f\|_\infty \int_{\mathbb{R}^n\setminus B_\delta} \rho_t(y)\,\mathrm{d}^n y \quad \leq \varepsilon.
\end{aligned}$$

To prove (ii) first assume that $f \in \mathcal{C}^0(\mathbb{R}^n)$ has compact support. Let $\varepsilon > 0$ be given. Then fix $K > 0$ such that $\operatorname{supp} f \subset B_K$ and

$$\int_{\mathbb{R}^n\setminus B_K} \rho_t \;\leq\; \tfrac{\varepsilon}{2\|f\|_p}.$$

Now consider the splitting

$$\big\|\rho_t * f - f\big\|_p^p \leq \int_{B_{2K}} \big|\rho_t * f - f\big|^p \;+ \int_{\mathbb{R}^n\setminus B_{2K}} \big|\rho_t * f - f\big|^p.$$

In the first term $|\rho_t * f - f| \to 0$ as $t \to 0$ converges uniformly on B_{2K} by (i), hence for sufficiently small $t > 0$ this term is smaller than $\frac{\varepsilon}{2}$.

The second term is estimated as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus B_{2K}} \left|\int_{\mathbb{R}^n} \rho_t(y)\big(f(x-y) - f(x)\big)\right|^p \mathrm{d}^n x \\
&\leq \int_{\mathbb{R}^n\setminus B_{2K}} \left(\int_{\mathbb{R}^n\setminus B_K} \rho_t(y)^{1-\frac{1}{p}} \rho_t(y)^{\frac{1}{p}} |f(x-y)| \mathrm{d}^n y\right)^p \mathrm{d}^n x \\
&\leq \int_{\mathbb{R}^n\setminus B_{2K}} \left(\int_{\mathbb{R}^n\setminus B_K} \rho_t\right)^{p-1} \int_{\mathbb{R}^n\setminus B_K} \rho_t(y)|f(x-y)|^p\,\mathrm{d}^n y \quad \mathrm{d}^n x \\
&= \int_{\mathbb{R}^n\setminus B_K} \rho_t(y) \int_{\mathbb{R}^n\setminus B_{2K}} |f(x-y)|^p\,\mathrm{d}^n x \quad \mathrm{d}^n y \\
&\leq \|f\|_p \int_{\mathbb{R}^n\setminus B_K} \rho_t(y)\,\mathrm{d}^n y \quad \leq \tfrac{\varepsilon}{2}.
\end{aligned}$$

Here we used the fact that for $|x| \geq 2K$ we have $f(x) = 0$, and if moreover $y \leq K$ then also $f(x-y) = 0$. Furthermore, we used the Hölder inequality as in (C.2). This proves (ii) for continuous functions f with compact support. But now for every $f \in L^p(\mathbb{R}^n)$ one can find a sequence f_i of such functions that converges to f in the L^p-norm. Then for all $t > 0$ and all $i \in \mathbb{N}$ use Young's inequality, theorem C.1, to obtain

$$\|\rho_t * f - f\|_p \leq \|\rho_t\|_1 \|f - f_i\|_p + \|\rho_t * f_i - f_i\|_p + \|f_i - f\|_p.$$

Here we have

$$\|\rho_t\|_1 \;=\; \int_{\mathbb{R}^n} \rho(\tfrac{x}{t})t^{-n}\,\mathrm{d}^n x \;=\; \int_{\mathbb{R}^n} \rho(z)\,\mathrm{d}^n z \;=\; 1.$$

So for given $\varepsilon > 0$ first fix $i \in \mathbb{N}$ such that $\|f - f_i\|_p \leq \frac{1}{4}\varepsilon$, then for all sufficiently small $t > 0$

$$\|\rho_t * f - f\|_p \;\leq\; \tfrac{1}{2}\varepsilon + \|\rho_t * f_i - f_i\|_p \;\leq\; \varepsilon.$$

□

Appendix D

The Dirichlet Problem

In this appendix we give a summary of results for the Dirichlet problem. There is good reference for this, so we only state the main theorems for use within this book and for comparison with the theorems for the Neumann problem.

Let M be an oriented compact Riemannian manifold with (possibly empty) boundary ∂M and denote by $\Delta = \mathrm{d}^*\mathrm{d}$ the Hodge Laplacian on functions. Then we consider the **Dirichlet boundary value problem**

$$\begin{cases} \Delta u = f & \text{on } M, \\ u = 0 & \text{on } \partial M. \end{cases} \tag{D.1}$$

A distribution $u \in \mathcal{D}(M)$ is called a **weak solution** of the Dirichlet problem for f if

$$\langle u \,,\, \Delta\psi \rangle = \langle f \,,\, \psi \rangle \qquad \forall \psi \in \mathcal{C}^\infty_\delta(M). \tag{D.2}$$

Here $\mathcal{C}^\infty_\delta(M)$ denotes the space of smooth functions on M that vanish on the boundary ∂M. For $u \in W^{2,1}(M)$ the concepts of strong and weak solution are equivalent.

In the following let $k \in \mathbb{N}_0$, $1 < p < \infty$, and $p^{-1} + (p^*)^{-1} = 1$. Denote by $W^{k,p}_\delta(M)$ the Sobolev space that is given by the closure of $\mathcal{C}^\infty_\delta(M)$ in the corresponding Sobolev norm. Then the basic existence and regularity results for the Dirichlet problem are the following.

Theorem D.1 *For every $f \in W^{k,p}(M)$ there exists a solution $u \in W^{k+2,p}(M)$ of (D.1), and this solution is unique.*

Theorem D.2 *Suppose that $u \in \mathcal{D}(M)$ is a weak solution of the Dirichlet problem (D.2) with $f \in W^{k,p}(M)$. Then $u \in W^{k+2,p}_\delta(M)$.*

Moreover, there there exists a constant C such that for all $u \in W^{k+2,p}_\delta(M)$

$$\|u\|_{W^{k+2,p}} \leq C\|\Delta u\|_{W^{k,p}}.$$

Theorem D.2$'$ *For all $k \in \mathbb{N}$ there exists a constant C such that the following holds: Suppose that $u \in \mathcal{D}(M)$ is a weak solution of the Dirichlet problem (D.2) for $f \in W^{-k,p}(M) = \big(W^{k,p^*}(M)\big)^*$. Then $u \in W^{-k+2,p}(M)$ and*

$$\|u\|_{W^{-k+2,p}} \leq C\|f\|_{W^{-k,p}}.$$

Remark D.3

(i) The optimal constant in theorem D.2 depends continuously on the metric of M with respect to the $W^{k+1,\infty}$-topology.

(ii) Theorem D.2$'$ with $k = 1$ provides a constant C such that the following holds: Let $u \in \mathcal{D}(M)$ and suppose that for some constant c_u

$$\big|\langle\, u\,,\, \Delta\psi\,\rangle\big| \leq c_u\|\psi\|_{W^{1,p^*}(M)} \qquad \forall \psi \in \mathcal{C}^\infty_\delta(M).$$

Then $u \in W^{1,p}_\delta(M)$ and $\|u\|_{W^{1,p}} \leq Cc_u$.

For domains in $\mathbb{R}^n$ and $k = 0$ these results can be found in [GT, Thm.9.14, 9.15]. The generalizations are proved by the same arguments that are used for the Neumann problem in part I. These results imply that the Laplace operator is a bijection from $W^{2,p}_\delta(M)$ to $L^p(M)$, hence:

Corollary D.4 *The Laplace operator $\Delta : W^{2,p}_\delta(M) \to L^p(M)$ is a Fredholm operator of index 0.*

Appendix E

Some Functional Analysis

In this appendix we state the functional analytic theorems that are most central and frequently used in this book. The references given are neither unique nor original but give the best presentation of the results that we could find.

Firstly, we state the implicit function theorem for general Banach spaces.

Theorem E.1 *[L, XIV,Thm.2.1]*
Let $D : X \times Y \to Z$ be a continuous map between Banach spaces that is differentiable with respect to Y, and let $(\alpha, \beta) \in X \times Y$ such that $D(\alpha, \beta) = 0$ and $\partial_Y D(\alpha, \beta)$ is bijective. Then there exist neighbourhoods $\mathcal{U} \subset X$ of α and $\mathcal{V} \subset Y$ of β and a continuous map $F : \mathcal{U} \to \mathcal{V}$ such that for every $x \in \mathcal{U}$ there is a unique $y \in \mathcal{V}$, namely $y = F(x)$, that solves $D(x, y) = 0$.

The following version of the Riesz representation theorem is called the Lax-Milgram theorem. It is fundamental for the theory of elliptic operators.

Theorem E.2 *[GT, Thm.5.8]*
Let $A : H \times H \to \mathbb{R}$ be a bounded, coercive bilinear form on a Hilbert space H, i.e. there exist constants K, $C > 0$ such that for all $x, y \in H$

$$|A(x, y)| \leq K \, \|x\| \cdot \|y\|, \qquad A(x, x) \geq C\|x\|^2.$$

Then for every bounded linear functional $F \in H^*$ there exists a unique element $u \in H$ such that

$$A(u, y) = F(y) \qquad \forall y \in H.$$

The Riesz representation theorem [GT, Thm.5.7] reduces this result to a statement, which is a simple consequence of the open mapping theorem: The bounded bilinear form $A : H \times H \to \mathbb{R}$ determines a bounded linear operator $T : H \to H$ with $\|T\| \leq K$. For $x \in H$ its value Tx is defined by

$$A(x, y) = \langle \, Tx \, , \, y \, \rangle \qquad \forall y \in H.$$

The assumption that A is coercive then becomes

$$\langle Tx\,,\,x\rangle \geq C\|x\|^2, \tag{E.1}$$

and the assertion of theorem E.2 is that every such operator T is bijective: For every $z \in H$ (determined by $F \in H^*$) there exists a unique $u \in H$ such that $Tu = z$ (which corresponds to $A(u,\cdot) = F$ in H^*).

The injectivity of T directly follows from (E.1): For all $x \in H$ we have $\|Tx\|\|x\| \geq C\|x\|^2$ and hence

$$\|Tx\| \geq C\|x\| \qquad \forall x \in H.$$

So T is injective, and by the below consequence of the open mapping theorem, lemma E.3 (ii), it also has a closed image. Moreover, (E.1) asserts that the dual operator T^* is injective: For all $x \in H$

$$\|T^*x\|\|x\| \;\geq\; \langle T^*x\,,\,x\rangle \;=\; \langle x\,,\,Tx\rangle \;\geq\; C\|x\|^2.$$

But now $(\operatorname{im} T)^\perp = \ker T^* = \{0\}$ and $\operatorname{im} T$ is closed, so $\operatorname{im} T = \{0\}^\perp = H$, and hence T also is surjective.

Here we have already used the following consequence of the open mapping theorem. This lemma also is an excellent trick to improve estimates.

Lemma E.3 *[Z, 3.12,Ex.3,Ex.4]*
Let $D : X \to Y$ be a bounded linear operator between Banach spaces.

(i) The following are equivalent:

a) D has a finite dimensional kernel and its image is closed.

b) There exists a compact operator $K : X \to Z$ to another Banach space Z and a constant C such that

$$\|u\|_X \leq C\left(\|Du\|_Y + \|Ku\|_Z\right) \qquad \forall u \in X. \tag{E.2}$$

(ii) The following are equivalent:

a) D is injective and its image is closed.

b) There exists a constant $\tilde{C}$ such that

$$\|u\|_X \leq \tilde{C}\|Du\|_Y \qquad \forall u \in X. \tag{E.3}$$

In particular, the combination of (i) and (ii) provides the following fact: If D satisfies (E.2) and is injective then in fact it satisfies (E.3).

Finally, here is a perturbation result that is very useful even in the case of linear operators on finite dimensional spaces, that is for matrices.

Lemma E.4 *Let $T : X \to Z$ and $S : X \to Z$ be bounded linear operators between Banach spaces. Suppose that T is bijective and $\|T^{-1}\| \cdot \|S\| < 1$. Then the perturbed operator $T + S : Z \to X$ also is bijective and*

$$\|(T+S)^{-1}\| \le \frac{\|T^{-1}\|}{1-\|T^{-1}\|\cdot\|S\|}, \qquad \|(T+S)^{-1} - T^{-1}\| \le \frac{\|T^{-1}\|^2\|S\|}{1-\|T^{-1}\|\cdot\|S\|}.$$

Proof: Firstly, T^{-1} is indeed bounded by lemma E.3 (ii). Then one simply checks

$$(T+S)^{-1} \;=\; \sum_{k=0}^{\infty}(-T^{-1}S)^k\, T^{-1} \;=\; \sum_{k=0}^{\infty} T^{-1}(-S\,T^{-1})^k.$$

Indeed, one has $\|T^{-1}S\| \le \|T^{-1}\| \cdot \|S\| < 1$, hence this geometric progression converges, and in fact it is the left and right inverse of $T + S$ since

$$\left(\sum_{k=0}^{N}(-T^{-1}S)^k\, T^{-1}\right)(T+S) \;=\; \mathbb{1} - (-T^{-1}S)^{N+1} \;\underset{N\to\infty}{\longrightarrow}\; \mathbb{1},$$

$$(T+S)\left(\sum_{k=0}^{N} T^{-1}(-S\,T^{-1})^k\right) \;=\; \mathbb{1} - (-S\,T^{-1})^{N+1} \;\underset{N\to\infty}{\longrightarrow}\; \mathbb{1}.$$

Moreover, again use the geometric progresssion for $\|T^{-1}\| \cdot \|S\| < 1$ to obtain

$$\|(T+S)^{-1}\| \;\le\; \sum_{k=0}^{\infty} \|T^{-1}\|^{k+1}\|S\|^k \;=\; \frac{\|T^{-1}\|}{1-\|T^{-1}\|\cdot\|S\|}$$

and hence

$$\begin{aligned}\|(T+S)^{-1} - T^{-1}\| &\le \|(T+S)^{-1}\| \cdot \|T-(T+S)\| \cdot \|T^{-1}\| \\ &\le \frac{\|T^{-1}\|^2\|S\|}{1-\|T^{-1}\|\cdot\|S\|}.\end{aligned}$$

□

In this book we often use the following particular case of this lemma : If the perturbed operator $\tilde{T} = T + S$ satisfies $\|\tilde{T} - T\| \le \frac{1}{2\|T^{-1}\|}$ (that is $\|T^{-1}\| \cdot \|S\| \le \frac{1}{2}$), then

$$\|\tilde{T}^{-1}\| \le 2\|T^{-1}\|, \qquad \|\tilde{T}^{-1} - T^{-1}\| \le 2\|T^{-1}\|^2\|\tilde{T} - T\|. \tag{E.4}$$

This is particularly useful since the constants in the estimates on $\tilde{T}^{-1}$ are independent of $\tilde{T}$.

List of Symbols

$\Delta = \mathrm{d}^*\mathrm{d}$, the (positive definite) Hodge Laplace operator on functions
∇_A : covariant derivative related to the connection A, see page 169
d_A : exterior differential related to the connection A, see page 170
d_A^* : formal adjoint of d_A, see page 173
$\langle \cdot, \cdot \rangle$: pairing of linear functionals with elements of the vector space or inner product, e.g. between differential forms, see pages 17,19,170,179
$\langle \cdot \wedge \cdot \rangle$: the wedge product on the level of differential forms, but the values are combined by an inner product, see page 172
$[\cdot \wedge \cdot]$: the wedge product on the level of differential forms, but the values are combined by the Lie bracket, see page 170
$\| \cdot \|_p$: L^p-norm, see page 180
$\| \cdot \|_\infty$: L^∞-norm, see page 181
$\mathcal{A}(P)$: space of smooth connections on a bundle P, see page 168
$\mathcal{A}^{k,p}(P)$: Sobolev space of connections on a bundle P, see page 180
$\mathcal{A}^{k,p}_{loc}(P)$: Sobolev space of connections on a noncompact bundle P, see page 181
$\mathcal{A}^{k,p}(U)$: Sobolev space of connections in a local trivialization over U, see page 181
$\mathcal{C}^\infty_\delta(M) = \left\{ \psi \in \mathcal{C}^\infty(M) \;\middle|\; \psi|_{\partial M} = 0 \right\}$
$\mathcal{C}^\infty_\nu(M) = \left\{ \psi \in \mathcal{C}^\infty(M) \;\middle|\; \frac{\partial \psi}{\partial \nu}|_{\partial M} = 0 \right\}$
$\mathcal{D}(M)$: space of distributions on M, see page 17
$\mathcal{E}(A) = \int_M |F_A|^q$ with $q \geq \frac{1}{2} \dim M$, see page 91
F_A : the curvature of a connection A, see page 170
G : (mostly) a compact Lie group
$\mathfrak{g}$: the Lie algebra of a Lie group G
$\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$, an associated bundle, see page 167
$\mathcal{G}(P)$: group of smooth gauge transformations on P, see page 167
$\mathcal{G}^{k,p}(P)$: Sobolev space of gauge transformations on a bundle P, see page 189
$\mathcal{G}^{k,p}_{loc}(P)$: Sobolev space of gauge transformations on a noncompact bundle P, see page 189

$\mathcal{G}^{k,p}(U)$: Sobolev space of gauge transformations in a local trivialization over U, see page 190
$\Gamma(P)$: space of smooth sections of a bundle P
$\mathbb{N} = \{1, 2, \ldots\}$
$\mathbb{N}_0 = \{0, 1, \ldots\}$
$W^{k,p}(M)$: Sobolev space of functions on M
$W^{k,\infty}(M)$: space of functions whose derivatives up to order k are bounded on M with the exception of a zero set; norm as on page 182
$W^{k,p}(M, E)$: Sobolev space of sections of a vector bundle $E \to M$, see page 180
$W^{k,p}(M, P)$: Sobolev space of sections of a fibre bundle $P \to M$, see page 185
$W^{0,2}(M) = L^2(M)$
$W^{-k,p}(M) = \big(W^{k,p^*}(M)\big)^*$, see pages 19,33
$W^{k,p}_{\partial}(M) = \big\{G|_{\partial M} \;\big|\; G \in W^{k+1,p}(M)\big\}$, see page 43
$\mathcal{YM}(A) = \int_M |F_A|^2$, see page 141

Index

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